Algebraic Phase Unwrapping with Self-Reciprocal Polynomial Algebra

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***Abstract*—Algebraic phase unwrapping gives the exact closed-form expression of the unwrapped phase of a complex polynomial. However, in computation of a Sturm sequence, there exist numerical instabilities due to coefficient growth. In this paper, we refine algebraic phase unwrapping by modifying the Sturm sequence with the newly defined self-reciprocal polynomial division. The proposed refinement enables us to compute the unwrapped phase, without suffering from the coefficient growth, by using the self-reciprocal subresultant which is newly defined as the determinant of a certain matrix. Numerical experiments show that algebraic phase unwrapping is greatly stabilized by the proposed method.**

I. Introduction

Phase unwrapping [1], [2] is a reconstruction problem of a continuous phase function. In many signal and image processing applications, the continuous phase function called the *unwrapped phase* relates to some physical quantity, e.g., the surface profile of an object in interferometry [3] or the degree of the magnetic field inhomogeneity in MRI [4]. However, only up to the principal value, *wrapped* into $(-\pi, \pi]$, of the continuous phase can be obtained at every sampling point. Therefore we must *unwrap* the samples of the *wrapped phase* to measure the physical quantity. This paper considers the following phase unwrapping problem for a univariate complex polynomial.

Problem 1 (Phase Unwrapping along the Unit Circle): Let $A(z) \in \mathbb{C}[z]$ satisfy $A_F(\omega) := A(e^{\imath\omega}) \neq 0$ for all $\omega \in [0, 2\pi]$. Then there exists a smooth function θ_{A_F} satisfying $A_F(\omega) = |A_F(\omega)|e^{\imath\theta_{A_F}(\omega)}$ for all $\omega \in [0, 2\pi]$, i.e., θ_{A_F} is the unwrapped phase of A along the unit circle. Compute θ_{A_F} at $\omega^* \in (0, 2\pi]$:

$$\theta_{A_F}(\omega^*) = \theta_{A_F}(0) + \int_0^{\omega^*} \Im\left[\frac{A_F'(\omega)}{A_F(\omega)}\right] \mathrm{d}\omega, \qquad (1)$$

where $\theta_{A_F}(0)$ is supposed to be given as the initial phase.

Problem 1 has to be solved for, e.g., evaluation of the stability of a certain digital filter [5] and computation of the *complex cepstrum* [6]. In Problem 1, since we can compute the value of $A_F(\omega)$ at any $\omega \in [0, 2\pi]$, the integral in (1) can be computed by using numerical integration techniques proposed, e.g., in [7], [8]. However, there is no guarantee that such numerical integration techniques give the exact unwrapped phase.

Algebraic phase unwrapping along the unit circle [9] gives the exact closed-form solution of Problem 1 (see Section II-B). The key of this algebraic solution is the computation of a certain *Sturm sequence*[1] by a polynomial division type algorithm.

This work was supported by JSPS Grants-in-Aid Grant Numbers 26-920 and B-15H02752 (e-mail: d-kita@fc.ritsumei.ac.jp; isao@sp.ce.titech.ac.jp).

[1]On the standard Sturm sequence, which is used to compute the number of zeros of a real polynomial on a given real interval, see, e.g., [10].

However, in the computations of the Sturm sequences, we encounter numerical instabilities due to *coefficient growth* which also occurs in the computation of the *greatest common divisor* for a pair of polynomials by the Euclidean algorithm [11]. As a result, especially for polynomials of high degree, thoughtless direct implementations of algebraic phase unwrapping sometimes cause the loss of the key property of the Sturm sequence, which leads to failure in phase unwrapping in the end.

In this paper, we stabilize algebraic phase unwrapping along the unit circle [9]. In Section III, after explaining typical numerical instabilities in the original polynomial division type algorithm (Algorithm 1), we newly define the *self-reciprocal polynomial division* (Theorem 1), in a way similar to the standard polynomial division, and generate a new Sturm sequence (Algorithm 2). The redefined Sturm sequence enables us to compute the unwrapped phase (Theorem 2) stably thanks to elimination of the conditional branch which causes *information loss* in Algorithm 1. Moreover, in Section IV, we newly define the *self-reciprocal subresultant*, as the determinant of a certain matrix, and present the relation between the signs in the Sturm sequence and those of the self-reciprocal subresultants (Theorem 3). Then, by replacing the inductive computations of the Sturm sequence with direct numerical evaluations of the self-reciprocal subresultants, we can compute the unwrapped phase without suffering from the coefficient growth. In Section V, numerical experiments exemplify the notable performance improvement achieved by the proposed stabilization techniques. Finally, in Section VI, we conclude this paper.

II. Preliminaries

A. Notation

Let $\mathbb{Z}_+$, $\mathbb{Z}_{++}$, $\mathbb{R}$, and $\mathbb{C}$ be the sets of all nonnegative integers, positive integers, real numbers, and complex numbers, respectively. We use $\imath \in \mathbb{C}$ to denote the imaginary unit, i.e., $\imath^2 = -1$. For any $c \in \mathbb{C}$, $\Im(c)$, $|c|$, and $\bar{c}$ respectively stand for the imaginary part, the magnitude, and the complex conjugate of c. For any $f : \mathbb{C} \to \mathbb{C}$, define $f_F : \mathbb{R} \ni \omega \mapsto f(e^{\imath\omega}) \in \mathbb{C}$. For any nonzero complex polynomial $C(z) = \sum_{k=l}^{m} c_k z^k \in \mathbb{C}[z]$ (s.t. $c_l c_m \neq 0$ and $m \geq l \geq 0$), define $\deg(C) := m$, $\mathrm{ldeg}(C) := l$, $\mathrm{cdeg}(C) := \frac{l+m}{2}$, $\mathrm{lc}(C) := c_m$, $\mathrm{mmc}(C) := \max\{|c_k|\}$, $C^*(z) := \sum_{k=l}^{m} \bar{c}_{l+m-k} z^k \in \mathbb{C}[z]$, and $C^\dagger(z) := z^{-\mathrm{cdeg}(C)}C(z) \in \mathbb{C}[z^{1/2}, z^{-1/2}]$. For the zero polynomial 0, we define $\deg(0) = -\infty$, $\mathrm{ldeg}(0) = \mathrm{cdeg}(0) = 0$, $\mathrm{lc}(0) = \mathrm{mmc}(C) = 0$, and $0^* = 0^\dagger = 0$. In particular, $C(z) \in \mathbb{C}[z]$ satisfying $C = C^*$ is called a *self-reciprocal polynomial* (or a *conjugate reciprocal polynomial*). If C is a self-reciprocal polynomial, then $C_F^\dagger$ is real-valued, i.e., $C_F^\dagger(\omega) \in \mathbb{R}$ for all $\omega \in \mathbb{R}$. For any set S, $\mathrm{card}(S)$ stands for its cardinal number. For $x \in \mathbb{R}$, its sign is defined by $\mathrm{sgn}(x) := x/|x|$ if $x \neq 0$ and $\mathrm{sgn}(x) := 0$ if $x = 0$, and $\arctan(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ denotes the principal value of the inverse tangent, i.e., $\tan(\arctan(x)) = x$.

B. Algebraic Phase Unwrapping along the Unit Circle I

In Problem 1, define self-reciprocal polynomials $A_{(0)}(z) := \frac{A(z)+A^*(z)}{2} = A^*_{(0)}(z) \in \mathbb{C}[z]$ and $A_{(1)}(z) := \frac{A(z)-A^*(z)}{2\imath} = A^*_{(1)}(z) \in \mathbb{C}[z]$. Then, the integral in (1) can be expressed as

$$\begin{aligned}\int_0^{\omega^*} \Im\left[\frac{A_F'(\omega)}{A_F(\omega)}\right] \mathrm{d}\omega &= \int_0^{\omega^*} \Im\left[\imath \mathrm{cdeg}(A) + \frac{A_F^{\dagger\prime}(\omega)}{A_F^\dagger(\omega)}\right] \mathrm{d}\omega \\ &= \mathrm{cdeg}(A)\omega^* \\ &\quad + \int_0^{\omega^*} \frac{A^{\dagger\prime}_{(1)F}(\omega)A^\dagger_{(0)F}(\omega) - A^\dagger_{(1)F}(\omega)A^{\dagger\prime}_{(0)F}(\omega)}{(A^\dagger_{(0)F}(\omega))^2 + (A^\dagger_{(1)F}(\omega))^2} \mathrm{d}\omega. \quad (2)\end{aligned}$$

If $A_{(0)} = 0$ or $A_{(1)} = 0$, then $\theta_{A_F}(\omega^*) = \theta_{A_F}(0) + \mathrm{cdeg}(A)\omega^*$. In what follows, we consider nonobvious cases where $A_{(0)} \neq 0$ and $A_{(1)} \neq 0$. Define the set of zeros of $A^\dagger_{(0)F}$ by

$$\begin{aligned}\mathcal{Z}^\dagger_{A_{(0)}} &:= \{\omega \in (0, 2\pi) \mid A^\dagger_{(0)F}(\omega) = 0\} \\ &= \begin{cases} \varnothing & \text{if } A^\dagger_{(0)F}(\omega) \neq 0, \ \forall \omega \in (0, 2\pi); \\ \{\nu_1, \nu_2, \ldots, \nu_z\} & \text{otherwise,} \end{cases}\end{aligned}$$

where $0 < \nu_1 < \nu_2 < \cdots < \nu_z < 2\pi$. By letting $\nu_0 := 0$, $\nu_k := \max(\{\nu_0, \nu_1, \ldots, \nu_z\} \cap [0, \omega^*))$ and $\mathcal{Q}^\dagger_A(\omega) := \frac{A^\dagger_{(1)F}(\omega)}{A^\dagger_{(0)F}(\omega)}$, the integral in (2) can be expressed as

$$\begin{aligned}&\int_0^{\omega^*} \frac{A^{\dagger\prime}_{(1)F}(\omega)A^\dagger_{(0)F}(\omega) - A^\dagger_{(1)F}(\omega)A^{\dagger\prime}_{(0)F}(\omega)}{(A^\dagger_{(0)F}(\omega))^2 + (A^\dagger_{(1)F}(\omega))^2} \mathrm{d}\omega \\ &= \sum_{i=0}^{k-1} \int_{\nu_i}^{\nu_{i+1}} [\arctan(\mathcal{Q}^\dagger_A(\omega))]' \,\mathrm{d}\omega + \int_{\nu_k}^{\omega^*} [\arctan(\mathcal{Q}^\dagger_A(\omega))]' \,\mathrm{d}\omega \\ &= -\lim_{\omega \to \nu_0 + 0} \arctan(\mathcal{Q}^\dagger_A(\omega)) + \lim_{\omega \to \omega^* - 0} \arctan(\mathcal{Q}^\dagger_A(\omega)) \\ &\quad + \sum_{i=1}^{k} \lim_{\substack{\omega_1 \to \nu_i - 0 \\ \omega_2 \to \nu_i + 0}} [\arctan(\mathcal{Q}^\dagger_A(\omega_1)) - \arctan(\mathcal{Q}^\dagger_A(\omega_2))] \\ &= -\lim_{\omega \to +0} \arctan(\mathcal{Q}^\dagger_A(\omega)) + \lim_{\omega \to \omega^* - 0} \arctan(\mathcal{Q}^\dagger_A(\omega)) \\ &\quad + \sum_{\nu_i \in (0, \omega^*)} \mathcal{X}^\dagger(\nu_i)\pi, \quad (3)\end{aligned}$$

where

$$\mathcal{X}^\dagger(\nu_i) := \begin{cases} +1 & \text{if } \begin{cases} \lim_{\omega \to \nu_i - 0} \mathcal{Q}^\dagger_A(\omega) = +\infty \text{ and} \\ \lim_{\omega \to \nu_i + 0} \mathcal{Q}^\dagger_A(\omega) = -\infty; \end{cases} \\ -1 & \text{if } \begin{cases} \lim_{\omega \to \nu_i - 0} \mathcal{Q}^\dagger_A(\omega) = -\infty \text{ and} \\ \lim_{\omega \to \nu_i + 0} \mathcal{Q}^\dagger_A(\omega) = +\infty; \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if $\mathcal{Z}^\dagger_{A_{(0)}}$ is known, (3) can be used to compute θ_{A_F}.

Even if $\mathcal{Z}^\dagger_{A_{(0)}}$ is unknown, algebraic phase unwrapping [9] can exactly compute θ_{A_F} without any numerical root finding or numerical integration technique. Define $B(z) := \frac{A(z)}{A(1)}$. Then, from $B_F(\omega) \neq 0$ for all $\omega \in [0, 2\pi]$, $B_F(0) = B_{(0)}(1) = 1$ and

$$\int_0^{\omega^*} \Im\left[\frac{A_F'(\omega)}{A_F(\omega)}\right] \mathrm{d}\omega = \int_0^{\omega^*} \Im\left[\frac{B_F'(\omega)}{B_F(\omega)}\right] \mathrm{d}\omega, \quad (4)$$

θ_{A_F} can be computed by Fact 1 below which was derived by extending the discovery of the direct relation between a real polynomial and its unwrapped phase along the unit circle [12].

Algorithm 1 Sturm Generating Algorithm I (SGA-I)

Input: $A_{(0)}(z) = A^*_{(0)}(z) \in \mathbb{C}[z]$, $A_{(1)}(z) = A^*_{(1)}(z) \in \mathbb{C}[z]$
Output: $(\Phi_k(\omega))_{k=0}^q$

$D_0(z) \leftarrow z^{-\mathrm{ldeg}(A_{(0)})} A_{(0)}(z)$
$D_1(z) \leftarrow z^{-\mathrm{ldeg}(A_{(1)})} (\frac{\imath}{z-1})^{o_1} A_{(1)}(z)$
(o_1 is the order of $z = 1$ as a zero of polynomial $A_{(1)}(z)$)
$\Phi_0(\omega) \leftarrow D_0^\dagger(e^{\imath\omega})$, $\Phi_1(\omega) \leftarrow D_1^\dagger(e^{\imath\omega})$
$k \leftarrow 1$
while $\deg(D_k) \geq 1$ **do**
  $\alpha_k \leftarrow \deg(D_{k-1}) - \deg(D_k)$
  $\beta_k \leftarrow \frac{\mathrm{lc}(D_{k-1})}{\mathrm{lc}(D_k)}$, $\gamma_k \leftarrow (-\imath)^{1-\alpha_k}\beta_k$
  $\widetilde{D}_{k+1}(z) \leftarrow \begin{cases} -D_{k-1}(z) + (\beta_k z^{\alpha_k} + \bar{\beta}_k) D_k(z) & \text{if } \alpha_k > 0 \\ -(\frac{z-1}{\imath})^{1-\alpha_k} D_{k-1}(z) + (\gamma_k z + \bar{\gamma}_k) D_k(z) & \text{if } \alpha_k \leq 0 \end{cases}$
  $D_{k+1}(z) \leftarrow z^{-\mathrm{ldeg}(\widetilde{D}_{k+1})} (\frac{\imath}{z-1})^{o_{k+1}} \widetilde{D}_{k+1}(z)$
  (o_{k+1} is the order of $z = 1$ as a zero of polynomial $\widetilde{D}_{k+1}(z)$)
  $\Phi_{k+1}(\omega) \leftarrow D^\dagger_{k+1}(e^{\imath\omega})$
  $k \leftarrow k + 1$
end while
$q \leftarrow \begin{cases} k & \text{if } \Phi_k \neq 0 \\ k-1 & \text{if } \Phi_k = 0 \end{cases}$

Fact 1 (Algebraic Phase Unwrapping along the Unit Circle I [9]): Let $A(z) = A_{(0)}(z) + \imath A_{(1)}(z) \in \mathbb{C}[z]$ satisfy $A_F(\omega) \neq 0$ for all $\omega \in [0, 2\pi]$, $A_F(0) = A_{(0)}(1) = 1$ and $A_{(1)} \neq 0$, where $A_{(0)}(z) := \frac{A(z)+A^*(z)}{2}$ and $A_{(1)}(z) := \frac{A(z)-A^*(z)}{2\imath}$. Let $(\Phi_k(\omega))_{k=0}^q$ be the sequence[2] of real-valued functions generated by applying Algorithm 1 (SGA-I) to $A_{(0)}$ and $A_{(1)}$. Define, at each $\omega \in [0, 2\pi]$,

$$\begin{aligned}V(\mathbf{\Phi}(\omega)) &:= V(\Phi_0(\omega), \Phi_1(\omega), \ldots, \Phi_q(\omega)) \\ &:= \mathrm{card}\{i \in [0, q-1] \mid \Phi_i(\omega)\Phi_{i+\varsigma_i(\omega)}(\omega) < 0\}\end{aligned}$$

as the number of sign changes in the entries of $\mathbf{\Phi}(\omega) \in \mathbb{R}^{q+1}$, where $\varsigma_i(\omega) := \min\{j \in \mathbb{Z}_{++} \mid \Phi_{i+j}(\omega) \neq 0\}$ (i.e., $V : \mathbb{R}^{q+1} \to \mathbb{Z}_+$ counts the number of sign changes by sequentially scanning $\mathbf{\Phi}(\omega)$ from left to right, and if there exists some Φ_k whose value at ω is $\Phi_k(\omega) = 0$, its sign is not counted). Then for every $\omega^* \in (0, 2\pi]$,

$$\begin{aligned}\int_0^{\omega^*} \Im\left[\frac{A_F'(\omega)}{A_F(\omega)}\right] \mathrm{d}\omega &= \mathrm{cdeg}(A)\omega^* \\ &+ \begin{cases} \arctan(\mathcal{Q}^\dagger_A(\omega^*)) + [V(\mathbf{\Phi}(\omega^*)) - V(\mathbf{\Phi}(0))]\pi & \\ \qquad\qquad \text{if } A^\dagger_{(0)F}(\omega^*) \neq 0; \\ \pi/2 + [V(\mathbf{\Phi}(\omega^*)) - V(\mathbf{\Phi}(0))]\pi \quad \text{if } A^\dagger_{(0)F}(\omega^*) = 0. \end{cases}\end{aligned} \quad (5)$$

[2] $(\Phi_k(\omega))_{k=0}^q$ is a *Sturm sequence* in the sense of [9, Theorem 5].

III. Refinement of Algebraic Phase Unwrapping by Elimination of Conditional Branch

A. Numerical Instabilities in Sturm Generating Algorithm

In the computation of the Sturm sequence $(\Phi_k(\omega))_{k=0}^q$ by Algorithm 1, the necessary number of digits to exactly express the rational coefficients of $\widetilde{D}_k$ and D_k grows very quickly. This phenomenon is the same as *coefficient growth* well-known in the computation of the *greatest common divisor* for a pair of polynomials by the Euclidean algorithm [11]. Therefore, in computer implementations of algebraic phase unwrapping with Algorithm 1, the coefficient growth causes the truncation error in the floating-point expression of the rational coefficients (or memory shortages due to the increase of the number of digits for the exact expression of the rational coefficients). In particular, once serious *information loss* (caused by the addition or the subtraction among numbers of ill-balanced absolute values) or *catastrophic cancellation* (caused by the subtraction among numbers of very close values) occurs, the gap between theoretical values and numerical values, in a digital computer, of $\boldsymbol{\Phi}(\omega)$ becomes unacceptably large. Unfortunately, Algorithm 1 often encounters information loss in the following situation.

Example 1 (Occurrence of Information Loss in SGA-I): Information loss almost always occurs by applying Algorithm 1 (SGA-I) to a pair of self-reciprocal polynomials $A_{(0)}$ and $A_{(1)}$ s.t. $|\deg(D_0) - \deg(D_1)|$ is relatively large. If $\deg(D_0) \ll \deg(D_1)$, i.e., $\alpha_1 = \deg(D_0) - \deg(D_1) \ll 0$, then the computation of $\widetilde{D}_2(z) = -(\frac{z-1}{\imath})^{1-\alpha_1} D_0(z) + (\gamma_1 z + \bar{\gamma}_1) D_1(z)$ causes information loss because the absolute values of the coefficients of $(\frac{z-1}{\imath})^{1-\alpha_1} = \sum_{l=0}^{1-\alpha_1} \frac{(1-\alpha_1)!}{l!(1-\alpha_1-l)!}(-1)^l \imath^{1-\alpha_1} z^l$ are very ill-balanced. If $\deg(D_0) \gg \deg(D_1)$, i.e., $\alpha_1 = \deg(D_0) - \deg(D_1) \gg 0$, then the computation of $\widetilde{D}_2(z) = -D_0(z) + (\beta_1 z^{\alpha_1} + \bar{\beta}_1) D_1(z)$ is usually stable. However, $\deg(D_1) \ll \deg(D_2)$, i.e., $\alpha_2 = \deg(D_1) - \deg(D_2) \ll 0$ ordinarily holds, and hence the computation of $\widetilde{D}_3(z) = -(\frac{z-1}{\imath})^{1-\alpha_2} D_1(z) + (\gamma_2 z + \bar{\gamma}_2) D_2(z)$ causes information loss.

Once serious information loss or catastrophic cancellation occurs, this inductively influences the process of Algorithm 1, which results in the loss of the central property of $(\Phi_k(\omega))_{k=0}^q$:

$$\Phi_k(\omega^*) = 0 \text{ at } \omega^* \in [0, 2\pi] \Rightarrow \Phi_{k-1}(\omega^*)\Phi_{k+1}(\omega^*) < 0 \quad (k = 1, 2, \ldots, q-1),$$

leading thus to failure in phase unwrapping based on (5). This situation restricts the practical applicability of algebraic phase unwrapping especially for polynomials of high degree.

B. Algebraic Phase Unwrapping along the Unit Circle II

At first, we newly define the *self-reciprocal polynomial division* in a way similar to the standard polynomial division. For a pair of nonzero polynomials $P_0(z) \in \mathbb{C}[z]$ and $P_1(z) \in \mathbb{C}[z]$, the standard polynomial division is expressed as

$$P_0(z) = Q(z)P_1(z) + R(z) \quad \text{s.t. } \deg(R) < \deg(P_1), \tag{6}$$

where the polynomial quotient $Q(z) \in \mathbb{C}[z]$ and the polynomial remainder $R(z) \in \mathbb{C}[z]$ are uniquely determined. Then, we consider, for a pair of nonzero self-reciprocal polynomials $P_0(z) = P_0^*(z) \in \mathbb{C}[z]$ and $P_1(z) = P_1^*(z) \in \mathbb{C}[z]$, whether or not there exist self-reciprocal polynomials $Q(z) = Q^*(z) \in \mathbb{C}[z]$ and $R(z) = R^*(z) \in \mathbb{C}[z]$ satisfying

$$P_0(z) = Q(z)P_1(z) + R(z) \quad \text{s.t. } \deg(R) - \mathrm{ldeg}(R) < \deg(P_1). \tag{7}$$

Algorithm 2 Sturm Generating Algorithm II (SGA-II)

Input: $A_{(0)}(z) = A_{(0)}^*(z) \in \mathbb{C}[z]$, $A_{(1)}(z) = A_{(1)}^*(z) \in \mathbb{C}[z]$
Output: $(\Phi_k(\omega))_{k=0}^q$

1: $\widetilde{D}_0(z) \leftarrow z^{-\mathrm{ldeg}(A_{(0)})}(\frac{\imath}{z-1})^{o_0} A_{(0)}(z)$
(o_0 is the order of $z = 1$ as a zero of polynomial $A_{(0)}(z)$)
2: $\widetilde{D}_1(z) \leftarrow z^{-\mathrm{ldeg}(A_{(1)})}(\frac{\imath}{z-1})^{o_1} A_{(1)}(z)$
(o_1 is the order of $z = 1$ as a zero of polynomial $A_{(1)}(z)$)
3: $\Phi_0(\omega) \leftarrow \widetilde{D}_0^\dagger(e^{\imath\omega})$, $\Phi_1(\omega) \leftarrow \widetilde{D}_1^\dagger(e^{\imath\omega})$
4: $(\delta_0, \delta_1) \leftarrow$
$$\begin{cases} (0,0) & \text{if } (\deg(\widetilde{D}_0) + \deg(\widetilde{D}_1)) \text{ is odd} \\ (1,0) & \text{if } (\deg(\widetilde{D}_0) + \deg(\widetilde{D}_1)) \text{ is even and } \deg(\widetilde{D}_0) \geq \deg(\widetilde{D}_1) \\ (0,1) & \text{if } (\deg(\widetilde{D}_0) + \deg(\widetilde{D}_1)) \text{ is even and } \deg(\widetilde{D}_1) > \deg(\widetilde{D}_0) \end{cases}$$
5: $D_0(z) \leftarrow (\frac{z-1}{\imath})^{\delta_0} \widetilde{D}_0(z)$, $D_1(z) \leftarrow (\frac{z-1}{\imath})^{\delta_1} \widetilde{D}_1(z)$
6: $k \leftarrow 1$
7: **while** $\deg(D_k) \geq 1$ **do**
8: $\quad \widetilde{D}_{k+1}(z) \leftarrow -D_{k-1}(z) - H_k(z)D_k(z)$
$\quad$($H_k(z) = H_k^*(z) \in \mathbb{C}[z]$, $\deg(\widetilde{D}_{k+1}) - \mathrm{ldeg}(\widetilde{D}_{k+1}) < \deg(D_k)$ and $\widetilde{D}_{k+1} \neq 0 \Rightarrow \mathrm{cdeg}(\widetilde{D}_{k+1}) = \mathrm{cdeg}(D_{k-1})$)
9: $\quad \Phi_{k+1}(\omega) \leftarrow \widetilde{D}_{k+1}^\dagger(e^{\imath\omega})$
10: $\quad D_{k+1}(z) \leftarrow z^{-\mathrm{ldeg}(\widetilde{D}_{k+1})} \widetilde{D}_{k+1}(z)$
11: $\quad k \leftarrow k + 1$
12: **end while**
13: $q \leftarrow \begin{cases} k & \text{if } \Phi_k \neq 0 \\ k-1 & \text{if } \Phi_k = 0 \end{cases}$

Theorem 1 (Self-Reciprocal Polynomial Division): Let P_0 and P_1 be nonzero self-reciprocal polynomials s.t. $\mathrm{ldeg}(P_0) = \mathrm{ldeg}(P_1) = 0$. If $\deg(P_0) < \deg(P_1)$ or $(\deg(P_0) + \deg(P_1))$ is odd, then there exist self-reciprocal polynomials Q and R satisfying (7). Moreover, if an additional condition $R \neq 0 \Rightarrow \mathrm{cdeg}(R) = \mathrm{cdeg}(P_0)$ is imposed, Q and R are uniquely determined, and $QR \neq 0 \Rightarrow (\deg(P_1) + \deg(R) - \mathrm{ldeg}(R))$ is odd.

We generate a new Sturm sequence $(\Phi_k(\omega))_{k=0}^q$ and self-reciprocal polynomial sequences $(\widetilde{D}_k(z))_{k=0}^q$ & $(D_k(z))_{k=0}^q$ by Algorithm 2 (SGA-II) which is based on Theorem 1 (see lines 4 and 8 in Algorithm 2). Differently from Algorithm 1, Algorithm 2 defines $\widetilde{D}_{k+1}$ $(k = 1, 2, \ldots, q-1)$ by using the self-reciprocal polynomial division without any conditional branch and rarely encounters serious information loss even if $|\deg(D_0) - \deg(D_1)|$ is relatively large.[3] Theorem 2 below is a refinement of Fact 1. (8) in Theorem 2 gives a direct solution of Problem 1 differently from (5) in Fact 1 where $B(z) = \frac{A(z)}{A(1)}$ and the relation in (4) are needed to solve Problem 1.

Theorem 2 (Algebraic Phase Unwrapping along the Unit Circle II): Let $A(z) = A_{(0)}(z) + \imath A_{(1)}(z) \in \mathbb{C}[z]$ satisfy $A_F(\omega) \neq 0$ for all $\omega \in [0, 2\pi]$, $A_{(0)} \neq 0$ and $A_{(1)} \neq 0$, where $A_{(0)}(z) := \frac{A(z) + A^*(z)}{2}$ and $A_{(1)}(z) := \frac{A(z) - A^*(z)}{2\imath}$.

[3]Specifically, the self-reciprocal polynomial division generates $\widetilde{D}_{k+1}$ by repeating computations similar to the upper branch on line 8 in Algorithm 1 until $\deg(\widetilde{D}_{k+1}) - \mathrm{ldeg}(\widetilde{D}_{k+1}) < \deg(D_k)$ is satisfied. Therefore, computations similar to the lower branch on line 8 in Algorithm 1 are not used.

Let $(\Phi_k(\omega))_{k=0}^{q}$ be the sequence of real-valued functions generated by applying Algorithm 2 (SGA-II) to $A_{(0)}$ and $A_{(1)}$. Then for every $\omega^* \in (0, 2\pi]$,

$$\begin{aligned}&\int_0^{\omega^*} \Im\left[\frac{A'_F(\omega)}{A_F(\omega)}\right] \mathrm{d}\omega \\ &= \operatorname{cdeg}(A)\omega^* - \begin{cases}\arctan(\mathcal{Q}_A^\dagger(0)) & \text{if } A_{(0)F}^\dagger(0) \neq 0;\\ \operatorname{sgn}(\Phi_0(0)\Phi_1(0))\pi/2 & \text{if } A_{(0)F}^\dagger(0) = 0;\end{cases} \\ &\quad + \begin{cases}\arctan(\mathcal{Q}_A^\dagger(\omega^*)) + [V(\boldsymbol{\Phi}(\omega^*)) - V(\boldsymbol{\Phi}(0))]\pi & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } A_{(0)F}^\dagger(\omega^*) \neq 0; & \\ \pi/2 + [V(\boldsymbol{\Phi}(\omega^*)) - V(\boldsymbol{\Phi}(0))]\pi \quad \text{if } A_{(0)F}^\dagger(\omega^*) = 0. & \end{cases}\end{aligned} \tag{8}$$

IV. Escape from Coefficient Growth with Self-Reciprocal Subresultant

Although the occurrence of information loss is greatly suppressed by the proposed algorithm (Algorithm 2), the coefficient growth still occurs in the computation of the redefined Sturm sequence. In [13], we stabilized algebraic phase unwrapping *along the real axis* [14] by replacing the standard polynomial division in (6) with the *subresultant* [10], [11]. In the following, we propose a similar technique for Theorem 2.

For a pair of nonzero self-reciprocal polynomials $P_0(z) = \sum_{k=0}^{m} a_k z^k = P_0^*(z)$ and $P_1(z) = \sum_{k=0}^{n} b_k z^k = P_1^*(z)$ s.t. $a_m = \bar{a}_0 \neq 0$, $b_n = \bar{b}_0 \neq 0$, $m > n \geq 1$ and $(m+n)$ is odd, we newly define the $\imath$th *self-reciprocal subresultant* $\mathrm{SRSres}_i[P_0, P_1](z) \in \mathbb{C}[z^{\frac{1}{2}}, z^{-\frac{1}{2}}]$ $(i = 0, 1, \ldots, n-1)$ by (9) where $|\cdot|$ denotes the determinant of a certain $(m+n-2i) \times (m+n-2i)$ matrix. Then we derive the following theorem.

Theorem 3 (Relation between the Signs in the Sturm Sequence and the Signs of the Self-Reciprocal Subresultants): Let $(\Phi_k(\omega))_{k=0}^{q}$ be the Sturm sequence generated by applying Algorithm 2 (SGA-II) to $A_{(0)} = A_{(0)}^* \neq 0$ and $A_{(1)} = A_{(1)}^* \neq 0$.

(i) If $\deg(D_0) > \deg(D_1) \geq 1$ and $\deg(D_{k+1}) = \deg(D_k) - 1$ for $k = 1, 2, \ldots, q-1$, then $q = \deg(D_1) + 1$ and

$$\begin{aligned}&\operatorname{sgn}(\Phi_k(\omega)) = (-1)^{(k-1)k/2 + (k-2)(\deg(D_0) - \deg(D_1) + k - 2)/2} \\ &\cdot \operatorname{sgn}\bigl(\mathrm{SRSres}_{\deg(D_1)-k+1}[D_0, D_1](e^{\imath\omega})\bigr) \quad (k = 2, 3, \ldots, q).\end{aligned}$$

(ii) If $\deg(D_1) > \deg(D_0) \geq 1$ and $\deg(D_{k+1}) = \deg(D_k) - 1$ for $k = 2, 3, \ldots, q-1$, then $\Phi_2 = -\Phi_0$, $q = \deg(D_0) + 2$ and

$$\begin{aligned}&\operatorname{sgn}(\Phi_k(\omega)) = (-1)^{(k-1)k/2 + (k-3)(\deg(D_1) - \deg(D_0) + k - 3)/2} \\ &\cdot \operatorname{sgn}\bigl(\mathrm{SRSres}_{\deg(D_0)-k+2}[D_1, D_0](e^{\imath\omega})\bigr) \quad (k = 3, 4, \ldots, q).\end{aligned}$$

Theorem 3 implies that we can evaluate the signs of $\Phi_k(\omega)$ $(k = 2, 3, \ldots, q)$ at $\omega \in [0, 2\pi]$ from $\mathrm{SRSres}_i[D_0, D_1](e^{\imath\omega})$ or $\mathrm{SRSres}_i[D_1, D_0](e^{\imath\omega})$, without computing the coefficients of $\widetilde{D}_k$ and D_k $(k = 2, 3, \ldots, q)$, as shown in Algorithm 3[4] since

$$\deg(D_{k+1}) = \deg(D_k) - 1 \quad (k = 1, 2, \ldots, q-1) \tag{10}$$

(or $k = 2, 3, \ldots, q-1$) holds almost always. As a result, we can compute $V(\boldsymbol{\Phi}(\omega^*))$ and $V(\boldsymbol{\Phi}(0))$ in (8), without suffering from the coefficient growth, by direct numerical evaluations of the self-reciprocal subresultants, i.e., the determinants in (9).

V. Numerical Experiments

We examine the numerical performance of algebraic phase unwrapping along the unit circle with Algorithms 1, 2, and 3 in the 64-bit floating point arithmetic. To make the situation likely to cause numerical instabilities, define a complex polynomial as $A(z) := A_{(0)}(z) + \imath A_{(1)}(z) \in \mathbb{C}[z]$ s.t. $A_{(k)}(z) := \widetilde{A}_{(k)}(z)\widehat{A}_{(k)}(z)/\mathrm{mmc}(\widetilde{A}_{(k)}\widehat{A}_{(k)}) = A_{(k)}^*(z)$ $(k = 0, 1)$, where $\widetilde{A}_{(0)}(z) := \frac{(z - e^{\imath 0.24\pi})(z - e^{\imath 0.76\pi})(z - e^{\imath 0.99\pi})(z - e^{\imath 1.26\pi})(z - e^{\imath 1.74\pi})}{e^{\imath(0.62+0.88+0.995+1.13+1.37)\pi}}$, $\widetilde{A}_{(1)}(z) := \frac{(z - e^{\imath 0.26\pi})(z - e^{\imath 0.74\pi})(z - e^{\imath 1.01\pi})(z - e^{\imath 1.24\pi})(z - e^{\imath 1.76\pi})}{e^{\imath(0.63+0.87+1.005+1.12+1.38)\pi}}$,

[4] Note that, in Algorithm 3, we suppose $\deg(D_0) \geq 1$ and $\deg(D_1) \geq 1$. This is because, if $\deg(D_0) = 0$ or $\deg(D_1) = 0$, then we can compute the unwrapped phase by using Algorithm 2 without any numerical instability.

$$\mathrm{SRSres}_i[P_0, P_1](z) := \left|\begin{array}{cccccccccccc}
a_m & a_{m-1} & \cdots & a_{m-n+i+2} & & a_{m-n+i+1} & a_0 & \cdots & a_{\frac{m-n+2i+3}{2}} & a_{\frac{m-n-1}{2}} & P_0^\dagger(z) z^{\frac{n-i-1}{2}} \\
 & a_m & \cdots & a_{m-n+i+3} & a_0 & a_{m-n+i+2} & a_1 & \cdots & a_{\frac{m-n+2i+5}{2}} & a_{\frac{m-n+1}{2}} & P_0^\dagger(z) z^{\frac{n-i-3}{2}} \\
 & & & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
 & & a_0 \cdots & a_m & a_{n-i-3} & a_{m-1} & a_{n-i-2} & \cdots & a_{\frac{m+n-1}{2}} & a_{\frac{m+n-2i-5}{2}} & P_0^\dagger(z) z^{-\frac{n-i-3}{2}} \\
 & a_0 & a_1 \cdots & & a_{n-i-2} & a_m & a_{n-i-1} & \cdots & a_{\frac{m+n+1}{2}} & a_{\frac{m+n-2i-3}{2}} & P_0^\dagger(z) z^{-\frac{n-i-1}{2}} \\
b_n & b_{n-1} & \cdots & b_{i+2} & & b_{i+1} & & \cdots & b_{\frac{n-m+2i+3}{2}} & & P_1^\dagger(z) z^{\frac{m-i-1}{2}} \\
 & b_n & \cdots & b_{i+3} & & b_{i+2} & & \cdots & b_{\frac{n-m+2i+5}{2}} & & P_1^\dagger(z) z^{\frac{m-i-3}{2}} \\
 & & \ddots & \vdots & & \vdots & & \cdots & \vdots & & \vdots \\
 & & & b_n & & b_{n-1} & & \cdots & \vdots & & \vdots \\
 & & & & & b_n & & \cdots & \vdots & & \vdots \\
 & & & & & & & \ddots & b_{i+2} & b_0 & P_1^\dagger(z) z^{\frac{n-i-2}{2}} \\
 & & & & & & & & \vdots & \vdots & \vdots \\
 & & & & & & & \iddots & b_n & b_{n-i-2} & P_1^\dagger(z) z^{-\frac{n-i-2}{2}} \\
 & & & & & & b_0 & \cdots & & \vdots & \vdots \\
 & & & & b_0 & & b_1 & \cdots & & \vdots & \vdots \\
 & & \iddots & & \vdots & & \vdots & \cdots & & \vdots & \vdots \\
 & & b_0 \cdots & & b_{n-i-3} & & b_{n-i-2} & \cdots & & b_{\frac{m+n-2i-5}{2}} & P_1^\dagger(z) z^{-\frac{m-i-3}{2}} \\
 & b_0 & b_1 \cdots & & b_{n-i-2} & & b_{n-i-1} & \cdots & & b_{\frac{m+n-2i-3}{2}} & P_1^\dagger(z) z^{-\frac{m-i-1}{2}}
\end{array}\right| \tag{9}$$

Algorithm 3 Sign Evaluation Under the Assumption in (10)

Input: $A_{(0)}(z) = A^*_{(0)}(z) \in \mathbb{C}[z]$, $A_{(1)}(z) = A^*_{(1)}(z) \in \mathbb{C}[z]$, $\omega^* \in [0, 2\pi]$

Output: $(\mathrm{sgn}(\Phi_k(\omega^*)))_{k=0}^{q}$

$\widetilde{D}_0(z) \leftarrow z^{-\mathrm{ldeg}(A_{(0)})}(\frac{\imath}{z-1})^{o_0} A_{(0)}(z)$

$\widetilde{D}_1(z) \leftarrow z^{-\mathrm{ldeg}(A_{(1)})}(\frac{\imath}{z-1})^{o_1} A_{(1)}(z)$

$\mathrm{sgn}(\Phi_0(\omega^*)) \leftarrow \mathrm{sgn}(\widetilde{D}_0^{\dagger}(e^{\imath\omega^*}))$, $\mathrm{sgn}(\Phi_1(\omega^*)) \leftarrow \mathrm{sgn}(\widetilde{D}_1^{\dagger}(e^{\imath\omega^*}))$

$$(\delta_0, \delta_1) \leftarrow \begin{cases} (0,0) & \text{if } (\deg(\widetilde{D}_0) + \deg(\widetilde{D}_1)) \text{ is odd} \\ (1,0) & \text{if } (\deg(\widetilde{D}_0) + \deg(\widetilde{D}_1)) \text{ is even and } \deg(\widetilde{D}_0) \geq \deg(\widetilde{D}_1) \\ (0,1) & \text{if } (\deg(\widetilde{D}_0) + \deg(\widetilde{D}_1)) \text{ is even and } \deg(\widetilde{D}_1) > \deg(\widetilde{D}_0) \end{cases}$$

$D_0(z) \leftarrow (\frac{z-1}{\imath})^{\delta_0} \widetilde{D}_0(z)$, $D_1(z) \leftarrow (\frac{z-1}{\imath})^{\delta_1} \widetilde{D}_1(z)$

$\deg_0 \leftarrow \deg(D_0)$, $\deg_1 \leftarrow \deg(D_1)$

if $\deg_0 > \deg_1$ **then**

  for $k = 2$ to $(\deg_1 + 1)$ **do**

    $\mathrm{sgn}(\Phi_k(\omega^*)) \leftarrow (-1)^{(k-1)k/2+(k-2)(\deg_0 - \deg_1 + k - 2)/2} \cdot \mathrm{sgn}(\mathrm{SRSres}_{\deg_1 - k+1}[D_0, D_1](e^{\imath\omega^*}))$

  end for

else

  $\mathrm{sgn}(\Phi_2(\omega^*)) \leftarrow -\mathrm{sgn}(\Phi_0(\omega^*))$

  for $k = 3$ to $(\deg_0 + 2)$ **do**

    $\mathrm{sgn}(\Phi_k(\omega^*)) \leftarrow (-1)^{(k-1)k/2+(k-3)(\deg_1 - \deg_0 + k - 3)/2} \cdot \mathrm{sgn}(\mathrm{SRSres}_{\deg_0 - k+2}[D_1, D_0](e^{\imath\omega^*}))$

end if

and $\widehat{A}_{(0)}(z) := \prod_{i=1}^{10} \frac{(z - r_i e^{\imath\phi_i})(z - e^{\imath\phi_i}/r_i)}{e^{\imath\phi_i}} \prod_{j=11}^{35} \frac{z - e^{\imath\phi_j}}{e^{\imath(\pi+\phi_j)/2}}$ & $\widehat{A}_{(1)}(z) := z^{15} \prod_{j=1}^{5} \frac{(z - \tilde{r}_j e^{\imath\psi_j})(z - e^{\imath\psi_j}/\tilde{r}_j)}{e^{\imath\psi_j}} \prod_{i=6}^{10} \frac{z - e^{\imath\psi_i}}{e^{\imath(\pi+\psi_i)/2}}$ are generated randomly.[5] In this situation, (i) serious information loss occurs in Algorithm 1 since $|\deg(D_0) - \deg(D_1)| = 30$ is relatively large, (ii) $A_{(0)}$ and $A_{(1)}$ have five pairs of close roots, i.e., $(e^{\imath 0.24\pi}, e^{\imath 0.26\pi})$, $(e^{\imath 0.76\pi}, e^{\imath 0.74\pi})$, $(e^{\imath 0.99\pi}, e^{\imath 1.01\pi})$, $(e^{\imath 1.26\pi}, e^{\imath 1.24\pi})$, and $(e^{\imath 1.74\pi}, e^{\imath 1.76\pi})$, which cause catastrophic cancellation in the computation of the greatest common divisor by the Euclidean algorithm [15], and (iii) we can verify whether algebraic phase unwrapping (Fact 1 or Theorem 2) succeeds or not since we know all roots of $A_{(0)}$ and hence the exact value of the unwrapped phase can be computed from (3).

Table I summarizes the results of algebraic phase unwrapping over $[0, 2\pi]$ in 1000 trials, where we can observe that the number of failures is reduced to less than $1/52$ and $1/111$ by replacing Algorithm 1 with Algorithms 2 and 3, respectively. Figure 1 depicts one example of the experimental results where Algorithms 1 and 2 fail in phase unwrapping at some points while Algorithm 3 succeeds in phase unwrapping over $[0, 2\pi]$.

VI. Conclusion

In this paper, we have refined algebraic phase unwrapping along the unit circle by modifying the Sturm sequence with the newly defined self-reciprocal polynomial division. The proposed modification can significantly stabilize algebraic phase unwrapping since the occurrence of information loss is greatly suppressed. Moreover, we clarified that the unwrapped phase can be computed, without suffering from the coefficient growth, by evaluating the signs in the Sturm sequence with the use of the newly defined self-reciprocal subresultants.

[5] $r_i, \tilde{r}_j \sim \mathcal{U}(0.6, 0.8)$, $\phi_i, \psi_j \sim \mathcal{U}(0, 2\pi)$ ($i \in [1, 10]$ and $j \in [1, 5]$), $\phi_i \sim \mathcal{U}(0.05\pi, 0.2\pi)$ ($i \in [11, 14]$), $\phi_i \sim \mathcal{U}(0.3\pi, 0.7\pi)$ ($i \in [15, 23]$), $\phi_i \sim \mathcal{U}(0.8\pi, 0.95\pi)$ ($i \in [24, 27]$), $\phi_i \sim \mathcal{U}(1.05\pi, 1.2\pi)$ ($i \in [28, 31]$), $\phi_i \sim \mathcal{U}(1.8\pi, 1.95\pi)$ ($i \in [32, 35]$), and $\psi_j \sim \mathcal{U}(1.3\pi, 1.7\pi)$ ($j \in [6, 10]$).

TABLE I
Performance Comparison for Random Complex Polynomials

Algorithm	Number of failures in phase unwrapping
Algorithm 1	892 (among 1000, in 64-bit floating point arithmetic)
Algorithm 2	17 (among 1000, in 64-bit floating point arithmetic)
Algorithm 3	8 (among 1000, in 64-bit floating point arithmetic)

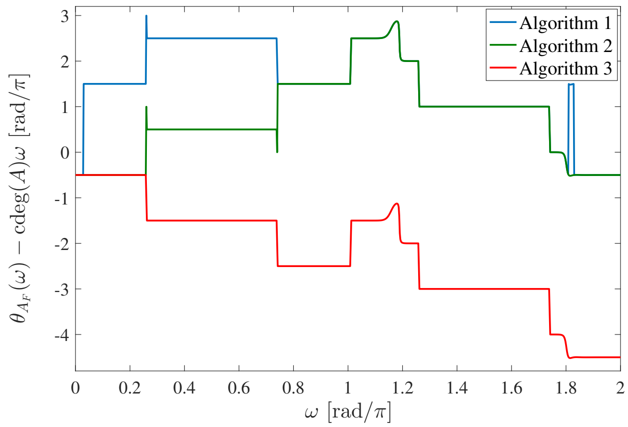

Fig. 1. Results of algebraic phase unwrapping with Algorithms 1, 2, and 3.

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