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Algebraic phase unwrapping along the real axis: extensions and stabilizations

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Abstract The unwrapped phase of a complex function is defined with a line integral of the gradient of the arctangent of the ratio of the real and imaginary parts of the function. The phase unwrapping, which is a problem to reconstruct the unwrapped phase of an unknown complex function from its finite observed samples, has been a key for estimating useful physical quantity in many signal and image processing applications. In the light of the functional data analysis, it is natural to estimate first the unknown complex function by a certain piecewise complex polynomial and then to compute the exact unwrapped phase of the piecewise complex polynomial with the algebraic phase unwrapping algorithms (Yamada et al. in IEEE Trans Signal Process 46(6), 1639–1664, 1998; Yamada and Bose in IEEE Trans Circuits Syst I Fundam Theory Appl 49(3), 298–304, 2002; Yamada and Oguchi in Multidimens Syst Signal Process 22(1–3), 191–211, 2011). In this paper, we propose several useful extensions and numerical stabilizations of the algebraic phase unwrapping along the real axis which was established originally in Yamada and Oguchi (Multidimens Syst Signal Process 22(1–3), 191–211, 2011). The proposed extensions include (i) removal of a certain critical assumption premised in the original algebraic phase unwrapping, and (ii) algebraic phase unwrapping for a pair of bivariate polynomials. Moreover, in order to resolve certain numerical instabilities caused by the coefficient growth in an inductive step in the original algorithm, we propose to compute directly a certain subresultant sequence without passing through the inductive step. The extensive numerical experiments exemplify the notable improvement, in the performance of the algebraic phase unwrapping, made by the proposed numerical stabilization.

Keywords Algebraic phase unwrapping · Two-dimensional phase unwrapping · Path independence condition · Numerical stabilization · Sturm sequence · Subresultant sequence

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1 Introduction

Suppose that

$$\left(d_{(0)}(\gamma(\zeta_k)), d_{(1)}(\gamma(\zeta_k))\right) := \left(f_{(0)}(\gamma(\zeta_k)) + \varepsilon_{(0)}(\gamma(\zeta_k)), f_{(1)}(\gamma(\zeta_k)) + \varepsilon_{(1)}(\gamma(\zeta_k))\right) \in \mathbb{R}^2$$

$(k = 1, 2, \ldots, s)$ are given as a finite sequence of 2-D noisy real vectors, where $f_{(i)} : \mathbb{R}^2 \to \mathbb{R}$ $(i = 0, 1)$ are unknown functions, $\varepsilon_{(i)} : \mathbb{R}^2 \to \mathbb{R}$ $(i = 0, 1)$ are additive random noise functions, and $\gamma : [a, b] \to \mathbb{R}^2$ is a known piecewise C^1 function which defines a path along the sample points $\gamma(\zeta_k) \in \mathbb{R}^2$ $(a \le \zeta_1 < \zeta_2 < \cdots < \zeta_s \le b)$.

For simplicity, denote by $F : [a, b] \ni t \mapsto F_{(0)}(t) + jF_{(1)}(t) \in \mathbb{C}$ a univariate complex valued function defined as

$$F_{(i)}(t) := f_{(i)}(\gamma(t)) \text{ for all } t \in [a, b] \quad (i = 0, 1).$$

The two-dimensional phase unwrapping of $(f_{(0)}, f_{(1)})$ along γ at $(x^*, y^*) := \gamma(t^*) \in \mathbb{R}^2$, or the phase unwrapping of F at $t^* \in [a, b]$ along the real axis, is a problem of estimating the unwrapped phase

$$\theta_f^{[\gamma]}(x^*, y^*) := \theta_F(t^*) := \theta_F(a) + \int_a^{t^*} \Im \left\{ \frac{F'_{(0)}(t) + jF'_{(1)}(t)}{F_{(0)}(t) + jF_{(1)}(t)} \right\} dt \tag{1}$$

by using the data $\left(d_{(0)}(\gamma(\zeta_k)), d_{(1)}(\gamma(\zeta_k))\right)$ $(k = 1, 2, \ldots, s)$, where we assume $\theta_F(a) \in (-\pi, \pi]$ is given as the initial phase satisfying $F(a) := |F(a)|e^{j\theta_F(a)} \neq 0$, the derivative $F'_{(i)}(t)$ of $F_{(i)}(t)$ $(i = 0, 1)$ are well-defined almost everywhere over $[a, b]$, and the integral in (1) is well-defined in the sense of *Lebesgue* (see, e.g., Rudin 1976, Chapter 11).

In many signal and image processing problems, the phase unwrapping has been a key for estimating some physical quantity (Ghiglia and Pritt 1998; Ying 2006), for example, surface topography in synthetic aperture radar (SAR) interferometry (Graham 1974; Zebker and Goldstein 1986; Goldstein et al. 1988; Jakowatz, Jr. et al. 1996) and synthetic aperture sonar (SAS) interferometry (Denbigh 1994; Hansen et al. 2003; Hayes and Gough 2009), wavefront distortion in adaptive optics (Fried 1977; Hudgin 1977; Noll 1978), the degree of magnetic field inhomogeneity in the water/fat separation problem of magnetic resonance imaging (MRI) (Glover and Schneider 1991; Szumowski et al. 1994; Moon-Ho Song et al. 1995), the relationship between the object phase and the bispectrum phase in astronomical imaging (Marron et al. 1990; Negrete-Regagnon 1996), the accurate profiling of mechanical parts by x-ray (Cloetens et al. 1999; Weitkamp et al. 2005) and the DOA estimation in array signal processing (Yamada and Oguchi 2011).

Despite the tremendous effort made so far, a technically reliable phase unwrapping has not yet been established for its practical use in wide range of signal and image processing. This is mainly because $\theta_F(t)$ $(a \le t \le b)$ is continuously defined along the arc $\gamma([a, b])$ as in (1) while most existing phase unwrapping algorithms, e.g., path-following methods (Goldstein et al. 1988; Judge and Bryanston-Cross 1994; Lin et al. 1994; Buckland et al. 1995), minimum-norm methods (Busbee et al. 1970; Pritt and Shipman 1994; Ghiglia and Romero 1996) and network flow methods (Flynn 1997; Costantini 1998) estimate the unwrapped phase θ_F only at ζ_k $(k = 1, 2, \ldots, s)$ without checking the consistency with θ_F at $t \in (\zeta_k, \zeta_{k+1})$.

In this paper, in the spirit of functional data analysis (Wahba 1990; Unser 1999; Ramsay and Silverman 2005; Schumaker 2007), we consider the situation where the functions $F_{(i)}$:

$[a,b] \to \mathbb{R}$ $(i=0,1)$ have been approximated respectively by known functions $\widetilde{F}_{(i)} : [a,b] \to \mathbb{R}$ $(i=0,1)$ through some smoothing techniques. In such a case, it is natural to estimate $\theta_F(t^*)$ in (1) by

$$\theta_{\widetilde{F}}(t^*) := \theta_{\widetilde{F}}(a) + \int_a^{t^*} \Im\left\{\frac{\widetilde{F}'_{(0)}(t)+j\widetilde{F}'_{(1)}(t)}{\widetilde{F}_{(0)}(t)+j\widetilde{F}_{(1)}(t)}\right\} dt. \tag{2}$$

In particular, motivated by the great success in the use of the Spline functions in the functional data analysis (Silverman 1985; Chui 1988; Wahba 1990; Unser 1999; Ramsay and Silverman 2005; Schumaker 2007), in this paper, we focus on a special case where $\widetilde{F}_{(i)} : [a,b] \to \mathbb{R}$ $(i=0,1)$ are given as piecewise polynomials (Kitahara and Yamada 2012). In this special case, by dividing the interval $[a,b]$ into finite subintervals if necessary, the computation of $\theta_{\widetilde{F}}(t^*)$ in (2) is reduced to the following phase unwrapping for a univariate complex polynomial along the real axis.

Problem 1 (*Phase unwrapping for a univariate complex polynomial along the real axis I*) For a given univariate complex polynomial $C(t) = \sum_{k=0}^m c_k t^k \in \mathbb{C}[t]$ satisfying $C(a) = |C(a)|e^{j\theta_C(a)} \neq 0$ with $\theta_C(a) \in (-\pi,\pi]$, let $C_{(0)}(t) := \sum_{k=0}^m \Re(c_k)t^k \in \mathbb{R}[t]$ and $C_{(1)}(t) := \sum_{k=0}^m \Im(c_k)t^k \in \mathbb{R}[t]$. Then compute the unwrapped phase of $C(t)$ at $t^* \in [a,b]$ by

$$\begin{aligned}\theta_C(t^*) &:= \theta_C(a) + \int_a^{t^*} \Im\left\{\frac{C'_{(0)}(t)+jC'_{(1)}(t)}{C_{(0)}(t)+jC_{(1)}(t)}\right\} dt,\\ &= \theta_C(a) + \int_a^{t^*} \Im\left\{\frac{B'_{(0)}(t)+jB'_{(1)}(t)}{B_{(0)}(t)+jB_{(1)}(t)}\right\} dt, \qquad (3)\\ &= \theta_C(a) + \int_a^{t^*} \frac{B'_{(1)}(t)B_{(0)}(t)-B_{(1)}(t)B'_{(0)}(t)}{\{B_{(0)}(t)\}^2+\{B_{(1)}(t)\}^2} dt, \qquad (4)\end{aligned}$$

where $B_{(0)}(t), B_{(1)}(t) \in \mathbb{R}[t]$ are respectively the real and imaginary parts of $B(t) := \frac{C(t)}{\mathrm{GCD}(C(t),\overline{C}(t))} = B_{(0)}(t) + jB_{(1)}(t) \in \mathbb{C}[t]$ for $\overline{C}(t) := \sum_{k=0}^m \bar{c}_k t^k \in \mathbb{C}[t]$, and GCD stands for the greatest common divisor.

(On the expression (3) of $\theta_C(t^*)$ and the integrability of (4), see Appendix 1).

Remark 1 (Possible inconsistency of θ_C caused by zero of $C(t)$) The function $\theta_C : [a,b] \to \mathbb{R}$ defined in (3) and (4) is always continuous. Moreover, even if there exists $t_z \in (a,b)$ satisfying $C(t_z) = 0$ and $C(t) \neq 0$ for all $t \in [a,t_z)$, θ_C satisfies $C(t) = |C(t)|e^{j\theta_C(t)}$ for all $t \in [a,t_z]$. However, in such a case, $C(t) = |C(t)|e^{j\theta_C(t)}$ is not necessarily guaranteed for $t \in (t_z,b]$. The possible inconsistency happens essentially by the same reason as the path dependency in the two-dimensional phase unwrapping (see Example 4 and Theorem 2).

Since $B(t) \neq 0$ for all $t \in \mathbb{R}$ is guaranteed in (3) and (4), it is sufficient to consider the following problem.

Problem 2 (*Phase unwrapping for a univariate complex polynomial along the real axis II*) For a given univariate complex polynomial $A(t) = \sum_{k=0}^m a_k t^k \in \mathbb{C}[t]$ satisfying $A(t) \neq 0$ for all $t \in [a,b]$, let $A_{(0)}(t) := \sum_{k=0}^m \Re(a_k)t^k \in \mathbb{R}[t]$ and $A_{(1)}(t) := \sum_{k=0}^m \Im(a_k)t^k \in \mathbb{R}[t]$. Then compute the unwrapped phase of $A(t)$ at $t^* \in [a,b]$ by

$$\theta_A(t^*) := \theta_A(a) + \int_a^{t^*} \frac{A'_{(1)}(t)A_{(0)}(t)-A_{(1)}(t)A'_{(0)}(t)}{\{A_{(0)}(t)\}^2+\{A_{(1)}(t)\}^2} dt, \tag{5}$$

where $\theta_A(a) \in (-\pi,\pi]$ satisfies $A(a) = |A(a)|e^{j\theta_A(a)}$.

The algebraic phase unwrapping for complex polynomials along the unit circle was established first in Yamada et al. (1998) by extending a first discovery (McGowan and Kuc 1982) of a direct relation between a real coefficient polynomial and its unwrapped phase along the unit circle. As its continuations, the algebraic phase unwrapping along the real axis (Yamada and Oguchi 2011), which is a first solution to Problem 2 (see Theorem 1 in Sect. 3.1), and that along the imaginary axis (Yamada and Bose 2002) have been developed. These methods do not require any numerical root finding or numerical integration technique. All the algorithms (Yamada et al. 1998; Yamada and Oguchi 2011; Yamada and Bose 2002) are essentially based on computing certain *general Sturm sequence* by polynomial division type algorithms. Potential application of the algebraic phase unwrapping is spanning widely in signal and image processing (Graham 1974; Fried 1977; Hudgin 1977; Noll 1978; Zebker and Goldstein 1986; Goldstein et al. 1988; Marron et al. 1990; Glover and Schneider 1991; Denbigh 1994; Szumowski et al. 1994; Moon-Ho Song et al. 1995; Jakowatz, Jr. et al. 1996; Negrete-Regagnon 1996; Cloetens et al. 1999; Hansen et al. 2003; Weitkamp et al. 2005; Hayes and Gough 2009; Yamada and Oguchi 2011) where reliable phase information has been demanded strongly.

However, in a direct computer implementation of all existing algorithms in Yamada et al. (1998), Yamada and Oguchi (2011) and Yamada and Bose (2002) as well as in a direct implementation of Algorithm 1 (Sturm-$\mathcal{R}$) in Sect. 3.1, we encounter numerical instabilities, especially for polynomials of relatively large degree, due to the unavoidable gap between theoretical value and numerical value computed by digital computer using finite digit number systems. Therefore, thoughtless direct implementation of the algebraic phase unwrapping algorithms for polynomials of large degree, sometimes results in the loss of key properties of the Sturm sequence. Such a loss leads to a certain serious failure of the phase unwrapping in the end.

The goal of this paper is to present several extensions and numerical stabilization of the algebraic phase unwrapping along the real axis (Yamada and Oguchi 2011). After giving preliminary results necessary in the later sections, we revisit the algebraic phase unwrapping along the real axis (Problem 2) in Sect. 3.1 where we present a new algorithm (Algorithm 1) to define a new Sturm sequence, unlike Yamada and Oguchi (2011, SGA 2), by eliminating the greatest factor $(t-a)^{e_i}$ from the first two polynomials $A_{(i)}(t)$ $(i=0,1)$ but not eliminating from the remaining polynomials generated by an inductive step. By this simplification, the Algorithm 1 (Sturm-$\mathcal{R}$) turns out to generate the *standard Sturm sequence* (Mishra 1993, Definition 8.4.2) for $\frac{A_{(0)}(t)}{(t-a)^{e_0}}$ and $\frac{A_{(1)}(t)}{(t-a)^{e_1}}$, and therefore we can express the new Sturm sequence with the *subresultant sequence* (Collins 1967; Brown and Traub 1971; Mishra 1993; Anai and Yokoyama 2011). Moreover, by this change for the first two polynomials, Theorem 1 based on Algorithm 1 can deal with a special case $A_{(0)}(a)=0$ which is excluded in Yamada and Oguchi (2011, Theorem 1). This relaxation is very useful especially in its application to a pair of piecewise polynomials (Sect. 3.2) because the value of $A_{(0)}(t)$ at $t=a$ is determined usually by the continuously connected polynomial defined on the adjacent subinterval. In Sect. 3.3, we consider the two-dimensional phase unwrapping and elucidate the condition for the path independence of the two-dimensional phase unwrapping. In particular, if bivariate polynomial functions $f_{(i)}:\mathbb{R}^2\to\mathbb{R}$ $(i=0,1)$ satisfy $f(x,y):=f_{(0)}(x,y)+jf_{(1)}(x,y)\neq 0$ for all (x,y) in a simply connected domain $D\subset\mathbb{R}^2$, we show, by Poincaré's lemma (see, e.g., Galbis and Maestre 2012), that the phase unwrapping along any piecewise C^1 arc $\gamma([a,b]):=\{\gamma(t)\in\mathbb{R}^2 \mid a\leq t\leq b\}\subset D$ defines uniquely a twice continuously differentiable function $\theta_f\in C^2(D)$, which is the two-dimensional unwrapped phase of f on D. The two-dimensional unwrapped phase θ_f can be computed with Algorithm 1 without requiring any numerical root finding or numerical integration technique. In Sect. 4, after starting with the observation of a typical situation causing numerical instabilities in the direct computer implementation of the algebraic phase unwrapping (Algorithm 1), in order to stabilize the computation of $\theta_A(t^*)$ in Theorem 1, we propose to replace the inductive computation of the polynomials $\Psi_k(t)$ $(k = 0,1,\ldots,q)$ in Algorithm 1, followed by their numerical evaluation at $t^* \in [a,b]$, with the direct numerical computation of the subresultant sequence (Collins 1967; Brown and Traub 1971) at t^*. For this purpose, we present complete relation between the sign of the Sturm sequence and that of the subresultant sequence (Propositions 4, 5 and Theorem 3). By the proposed replacement, the sign of the ideal standard Sturm sequence can be computed without suffering from the propagation of errors caused by the coefficient growth in the process of Algorithm 1, and then the algebraic phase unwrapping is stabilized greatly even for polynomials of relatively large degree. The extensive numerical experiments, of the algebraic phase unwrapping along the real axis, exemplify the notable performance improvement made by the proposed numerical stabilization.

The proposal in this paper is expected to be a firm mathematical foundation for wider application of the algebraic phase unwrapping, e.g., in a combination with the spline smoothing (Silverman 1985; Chui 1988; Wahba 1990; Unser 1999; Ramsay and Silverman 2005; Schumaker 2007), to practical signal and image processing problems.

2 Preliminaries

2.1 Notation

Let $\mathbb{N}^*$, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the set of all positive integers, real numbers and complex numbers. We use $j \in \mathbb{C}$ to denote the imaginary unit satisfying $j^2 = -1$. For any $c \in \mathbb{C}$, $\Re(c)$, $\Im(c)$ and $\bar{c}$ stand respectively for the real part, the imaginary part and the complex conjugate of c. For any $C(t) = \sum_{k=0}^{m} c_k t^k \in \mathbb{C}[t]$ (s.t. $c_m \neq 0$ and $m \geq 0$), we define $\overline{C}(t) := \sum_{k=0}^{m} \bar{c}_k t^k \in \mathbb{C}[t]$, $\deg(C) := m$, $\mathrm{lc}(C) := c_m$ and $\mathrm{mmc}(C) := \max\{|c_0|, |c_1|, \ldots, |c_m|\}$. The degree of the zero polynomial is defined to be $-\infty$. For any $C(t) = \sum_{k=0}^{m} c_k t^k \in \mathbb{C}[t]$, we use the expression $C(t) = C_{(0)}(t) + jC_{(1)}(t)$, where $C_{(0)}(t) := \sum_{k=0}^{m} \Re(c_k) t^k \in \mathbb{R}[t]$ and $C_{(1)}(t) := \sum_{k=0}^{m} \Im(c_k) t^k \in \mathbb{R}[t]$. For any $x \in \mathbb{R}$, its sign is defined by

$$\mathrm{sgn}(x) := \begin{cases} x/|x| & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

and arctan denotes the principle value inverse tangent satisfying $\tan(\arctan(x)) = x$ and $-\frac{\pi}{2} < \arctan(x) < \frac{\pi}{2}$.

2.2 Elementary facts on vector calculus

For the discussion on the continuity of the two-dimensional unwrapped phase and the applicability of the algebraic phase unwrapping to two-dimensional case (Sect. 3.3), we need the following classical results (see, e.g., Apostol 1974, Rudin 1976, Galbis and Maestre 2012).

Fact 1 (Green's theorem and Poincaré's lemma)

(a) (Green's theorem) *Suppose U is an open set in $\mathbb{R}^2$, $P, Q \in C^1(U)$, i.e., $P : U \to \mathbb{R}$ and $Q : U \to \mathbb{R}$ are continuously differentiable over U, and Ω is a closed subset of U, with positively oriented boundary $\partial\Omega$. Then we have*

$$\oint_{\partial\Omega} [P(x,y)dx + Q(x,y)dy] = \iint_{\Omega} \left(\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y) \right) dxdy.$$

(b) (Poincaré's lemma: Condition for exact differentiability) *Suppose D is a simply connected domain in $\mathbb{R}^2$, $P, Q \in C^1(D)$, and $\frac{\partial P}{\partial y}(x,y) = \frac{\partial Q}{\partial x}(x,y)$ for all $(x,y) \in D$. Then there exists a function $f \in C^2(D)$ satisfying*

$$\frac{\partial f}{\partial x}(x,y) = P(x,y) \ \text{ and } \ \frac{\partial f}{\partial y}(x,y) = Q(x,y) \ \text{ for all } (x,y) \in D.$$

The function $f \in C^2(D)$ is nothing but the scalar potential of the vector field $(P(x,y), Q(x,y))$ over D.

2.3 Subresultant and polynomial remainder

For a pair of real polynomials

$$\begin{aligned} P_0(t) &:= a_m t^m + a_{m-1} t^{m-1} + \cdots + a_1 t + a_0, \\ P_1(t) &:= b_n t^n + b_{n-1} t^{n-1} + \cdots + b_1 t + b_0, \end{aligned}$$

s.t. $a_m \neq 0$ and $b_n \neq 0$, define $R_i(P_0, P_1, t) \in \mathbb{R}[t]^{(m+n-2i)\times(m+n-2i)}$ $(i = 0, 1, \ldots, \min\{m-1, n-1\})$ by

$$R_i(P_0,P_1,t) := \overbrace{\left(\begin{array}{cccccccccccc}
a_m & a_{m-1} & \cdots & a_i & a_{i-1} & \cdots & a_0 & & & & P_0(t)t^{n-i-1} \\
& a_m & a_{m-1} & \cdots & a_i & a_{i-1} & \cdots & a_0 & & & P_0(t)t^{n-i-2} \\
& & \ddots & \ddots & & \ddots & \ddots & & \ddots & & \vdots \\
& & & a_m & a_{m-1} & \cdots & a_i & a_{i-1} & \cdots & a_0 & P_0(t)t^{i+1} \\
& & & & a_m & a_{m-1} & \cdots & a_i & \cdots & a_1 & P_0(t)t^{i} \\
& & & & & \ddots & \ddots & & \ddots & \vdots & \vdots \\
& & & & & & a_m & a_{m-1} & \cdots & a_i & P_0(t)t \\
& & & & & & & a_m & \cdots & a_{i+1} & P_0(t) \\
b_n & b_{n-1} & \cdots & b_i & b_{i-1} & \cdots & b_0 & & & & P_1(t)t^{m-i-1} \\
& b_n & b_{n-1} & \cdots & b_i & b_{i-1} & \cdots & b_0 & & & P_1(t)t^{m-i-2} \\
& & \ddots & \ddots & & \ddots & \ddots & & \ddots & & \vdots \\
& & & b_n & b_{n-1} & \cdots & b_i & b_{i-1} & \cdots & b_0 & P_1(t)t^{i+1} \\
& & & & b_n & b_{n-1} & \cdots & b_i & \cdots & b_1 & P_1(t)t^{i} \\
& & & & & \ddots & \ddots & & \ddots & \vdots & \vdots \\
& & & & & & b_n & b_{n-1} & \cdots & b_i & P_1(t)t \\
& & & & & & & b_n & \cdots & b_{i+1} & P_1(t)
\end{array}\right)}^{m+n-2i}
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right\} n-i \\
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right\} m-i
\end{array}.$$

Then the ith *subresultant* $\mathrm{Sres}_i(P_0, P_1, t)$ of $P_0(t)$ and $P_1(t)$ is defined as the determinant of $R_i(P_0, P_1, t)$, i.e.,

$$\mathrm{Sres}_i(P_0,P_1,t) := \det(R_i(P_0,P_1,t)) \in \mathbb{R}[t] \quad (i = 0, 1, \ldots, \min\{m-1, n-1\}).$$

It is well-known (Collins 1967; Brown and Traub 1971; Anai and Yokoyama 2011) that the degree of the ith subresultant does not exceed i. For $P_0(t)$ and $P_1(t)$, we also define $M_i(P_0, P_1) \in \mathbb{R}^{(m+n-2i)\times(m+n-2i)}$ $(i = 0, 1, \ldots, \min\{m-1, n-1\})$ by

$$
M_i(P_0,P_1) := \overbrace{\begin{pmatrix}
a_m & a_{m-1} & \cdots & a_i & a_{i-1} & \cdots & a_0 & & & & \\
 & a_m & a_{m-1} & \cdots & a_i & a_{i-1} & \cdots & a_0 & & & \\
 & & \ddots & \ddots & & \ddots & \ddots & & \ddots & & \\
 & & & a_m & a_{m-1} & \cdots & a_i & a_{i-1} & \cdots & a_0 & \\
 & & & & a_m & a_{m-1} & \cdots & a_i & \cdots & a_1 & a_0 \\
 & & & & & \ddots & \ddots & & \ddots & \vdots & \vdots \\
 & & & & & & a_m & a_{m-1} & \cdots & a_i & a_{i-1} \\
 & & & & & & & a_m & \cdots & a_{i+1} & a_i \\
b_n & b_{n-1} & \cdots & b_i & b_{i-1} & \cdots & b_0 & & & & \\
 & b_n & b_{n-1} & \cdots & b_i & b_{i-1} & \cdots & b_0 & & & \\
 & & \ddots & \ddots & & \ddots & \ddots & & \ddots & & \\
 & & & b_n & b_{n-1} & \cdots & b_i & b_{i-1} & \cdots & b_0 & \\
 & & & & b_n & b_{n-1} & \cdots & b_i & \cdots & b_1 & b_0 \\
 & & & & & \ddots & \ddots & & \ddots & \vdots & \vdots \\
 & & & & & & b_n & b_{n-1} & \cdots & b_i & b_{i-1} \\
 & & & & & & & b_n & \cdots & b_{i+1} & b_i
\end{pmatrix}}^{m+n-2i}
\begin{array}{l}
\left.\vphantom{\begin{matrix}a\\a\\a\\a\\a\\a\\a\\a\end{matrix}}\right\} n-i \\
\left.\vphantom{\begin{matrix}b\\b\\b\\b\\b\\b\\b\\b\end{matrix}}\right\} m-i
\end{array}.
$$

$\det(M_i(P_0,P_1))$ is called the *principal subresultant coefficient* and satisfies

$$
\det(M_i(P_0,P_1)) = \begin{cases} \mathrm{lc}(\mathrm{Sres}_i(P_0,P_1,t)) \neq 0 & \text{if } \deg(\mathrm{Sres}_i(P_0,P_1,t)) = i, \\ 0 & \text{if } \deg(\mathrm{Sres}_i(P_0,P_1,t)) < i. \end{cases} \tag{6}
$$

In particular, $M_0(P_0,P_1)$ is the *Sylvester matrix* of $P_0(t)$ and $P_1(t)$, and $\det(M_0(P_0,P_1)) = \mathrm{Sres}_0(P_0,P_1,t)$ is the *resultant* of $P_0(t)$ and $P_1(t)$.

The subresultant is closely-linked to the polynomial remainder as follows.

Fact 2 (Relation between the subresultant and the polynomial remainder (Brown and Traub 1971, Lemma 1)) *Let $P_{k-1}(t)$, $P_k(t)$ and $P_{k+1}(t)$ be nonzero polynomials satisfying*

$$
\left.\begin{array}{c} \deg(P_{k-1}) \geq \deg(P_k) > \deg(P_{k+1}) \\ P_{k+1}(t) := P_{k-1}(t) - Q_k(t)P_k(t) \textit{ with } Q_k(t) \in \mathbb{R}[t] \end{array}\right\}.
$$

Then, the ith subresultant $\mathrm{Sres}_i(P_{k-1},P_k,t)$ $(i = 0,1,\ldots,\deg(P_k)-1)$ *can be expressed as*

$$
\mathrm{Sres}_i(P_{k-1},P_k,t) = \begin{cases}
(-1)^{\deg(P_{k-1})-\deg(P_k)+1}(\mathrm{lc}(P_k))^{\deg(P_{k-1})-\deg(P_k)+1}P_{k+1}(t) \\
\qquad \textit{for } i = \deg(P_k)-1, \\
0 \quad \textit{for } i \in [\deg(P_{k+1})+1, \deg(P_k)-2] \;\; (\textit{if } \deg(P_{k+1}) < \deg(P_k)-2), \\
(-1)^{(\deg(P_{k-1})-\deg(P_k)+1)(\deg(P_k)-\deg(P_{k+1}))}(\mathrm{lc}(P_k))^{\deg(P_{k-1})-\deg(P_{k+1})} \\
\;\times (\mathrm{lc}(P_{k+1}))^{\deg(P_k)-\deg(P_{k+1})-1}P_{k+1}(t) \\
\qquad \textit{for } i = \deg(P_{k+1}), \\
(-1)^{(\deg(P_{k-1})-\deg(P_k)+1)(\deg(P_k)-i)}(\mathrm{lc}(P_k))^{\deg(P_{k-1})-\deg(P_{k+1})} \\
\;\times \mathrm{Sres}_i(P_k,P_{k+1},t) \\
\qquad \textit{for } i \in [0, \deg(P_{k+1})-1] \;\; (\textit{if } \deg(P_{k+1}) \geq 1).
\end{cases}
$$

Remark 2 (The equivalence between Fact 2 and Brown and Traub 1971, Lemma 1) In Brown and Traub (1971, Lemma 1), $\mathrm{Sres}_i(P_{k-1}, P_k, t)$ $(i \in [0, \deg(P_{k+1}) - 1])$ is expressed as

$$\mathrm{Sres}_i(P_{k-1}, P_k, t) = (-1)^{(\deg(P_{k-1})-i)(\deg(P_k)-i)}(\mathrm{lc}(P_k))^{\deg(P_{k-1})-\deg(P_{k+1})}\mathrm{Sres}_i(P_k, P_{k+1}, t),$$

which is seemingly different from the expression in Fact 2. The equivalence between these expressions is verified as follows.

If $(\deg(P_k) - i)$ is even, we have

$$(-1)^{(\deg(P_{k-1})-\deg(P_k)+1)(\deg(P_k)-i)} = (-1)^{(\deg(P_{k-1})-i)(\deg(P_k)-i)} = 1.$$

If $(\deg(P_k) - i)$ is odd, i.e., $\deg(P_k) - i = 2\rho - 1$ (for some $\rho \in \mathbb{N}^*$), we have

$$\begin{aligned}(-1)^{(\deg(P_{k-1})-\deg(P_k)+1)(\deg(P_k)-i)} &= (-1)^{(\deg(P_{k-1})-i-(2\rho-2))(\deg(P_k)-i)} \\ &= (-1)^{(\deg(P_{k-1})-i)(\deg(P_k)-i)}.\end{aligned}$$

2.4 Expression of unwrapped phase

The next proposition is a slight extension of Yamada and Oguchi (2011, Proposition 2). The expression (7) gives a useful formula to compute the exact unwrapped phase $\theta_A(t^*)$ when all distinct real roots, in (a,b), of $A_{(0)}(t)$ are known.

Proposition 1 (An expression of the unwrapped phase) *Let $A(t) := A_{(0)}(t) + jA_{(1)}(t) \in \mathbb{C}[t]$ satisfy $A(t) \neq 0$ for all $t \in [a,b]$. If $A_{(0)}(t) \equiv 0$ or $A_{(1)}(t) \equiv 0$, we have, from (5), $\theta_A(t^*) = \theta_A(a)$ for all $t^* \in [a,b]$. Otherwise, define*

$$\begin{aligned}\mathcal{Z}_{A_{(0)}} &:= \{t \in (a,b) \mid A_{(0)}(t) = 0\} \\ &= \begin{cases} \varnothing & \text{if } A_{(0)}(t) \neq 0 \text{ for all } t \in (a,b), \\ \{\mu_1, \mu_2, \ldots, \mu_z\} & \text{otherwise,} \end{cases}\end{aligned}$$

where $a < \mu_1 < \mu_2 < \cdots < \mu_z < b$, and

$$\mathcal{X}(\mu_i) := \begin{cases} +1 & \text{if } \begin{cases} A_{(0)}(t)A_{(1)}(t) > 0 & \text{for } t \in (\mu_i - \varepsilon, \mu_i) \text{ and} \\ A_{(0)}(t)A_{(1)}(t) < 0 & \text{for } t \in (\mu_i, \mu_i + \varepsilon), \end{cases} \\ -1 & \text{if } \begin{cases} A_{(0)}(t)A_{(1)}(t) < 0 & \text{for } t \in (\mu_i - \varepsilon, \mu_i) \text{ and} \\ A_{(0)}(t)A_{(1)}(t) > 0 & \text{for } t \in (\mu_i, \mu_i + \varepsilon), \end{cases} \\ 0 & \text{otherwise,} \end{cases}$$

for μ_i $(i = 1, 2, \ldots, z)$ and for sufficiently small $\varepsilon > 0$. Then we have, for any $t^* \in (a,b]$,

$$\theta_A(t^*) = \theta_A(a) - \lim_{t \to a+0} \arctan\{\mathcal{Q}_A(t)\} + \lim_{t \to t^*-0} \arctan\{\mathcal{Q}_A(t)\} + \Lambda(t^*)\pi, \quad (7)$$

where $\mathcal{Q}_A(t) := \dfrac{\Im\{A(t)\}}{\Re\{A(t)\}} = \dfrac{A_{(1)}(t)}{A_{(0)}(t)}$ *and* $\Lambda(t^*) := \displaystyle\sum_{\mu_i \in (a, t^*)} \mathcal{X}(\mu_i).$

Proof See Appendix 2.

3 Extensions of the algebraic phase unwrapping along the real axis

3.1 Relaxation of the conditions for algebraic phase unwrapping

Under the assumptions on $A_{(0)}(t), A_{(1)}(t) \in \mathbb{R}[t]$, in Proposition 1, including $A_{(0)}(t) \not\equiv 0$ and $A_{(1)}(t) \not\equiv 0$, define the sequence of real polynomials $\{\Psi_k(t)\}_{k=0}^{q}$ by applying Algorithm 1 (Sturm-$\mathcal{R}$) to $A_{(0)}(t)$ and $A_{(1)}(t)$. The sequence $\{\Psi_k(t)\}_{k=0}^{q}$ is a *Sturm sequence* in the sense of Lemma 1. Note that, in comparison to Yamada and Oguchi (2011, SGA 2), Algorithm 1 differently defines $\{\Psi_k(t)\}_{k=0}^{q}$, i.e., $\Psi_0(t)$ and $\Psi_1(t)$ are defined respectively by eliminating the greatest factor $(t-a)^{e_0}$ and $(t-a)^{e_1}$ from $A_{(0)}(t)$ and $A_{(1)}(t)$, but $\Psi_k(t)$ $(k = 2, 3, \ldots, q)$ are defined without eliminating the greatest factor $(t-a)^{e_k}$.

Algorithm 1 Sturm generating algorithm along the real axis (Sturm-$\mathcal{R}$)

Input: $A_{(0)}(t), A_{(1)}(t) \in \mathbb{R}[t]$ and $a, b \in \mathbb{R}$ (s.t. $A_{(0)}(t) + jA_{(1)}(t) \neq 0$ for all $t \in [a,b]$ and $A_{(0)}(t), A_{(1)}(t) \not\equiv 0$)

1: $\Psi_0(t) \leftarrow \dfrac{A_{(0)}(t)}{(t-a)^{e_0}}$, $\Psi_1(t) \leftarrow \dfrac{A_{(1)}(t)}{(t-a)^{e_1}}$
(where e_i denotes the order of $t = a$ as a zero of polynomial $A_{(i)}(t)$ $(i = 0, 1)$)
2: $k \leftarrow 1$
3: **while** $\deg(\Psi_k) \neq 0$ **do**
4: $\quad \Psi_{k+1}(t) \leftarrow -\Psi_{k-1}(t) - H_k(t)\Psi_k(t)$ (where $H_k(t) \in \mathbb{R}[t]$ and $\deg(\Psi_{k+1}) < \deg(\Psi_k)$)
5: $\quad k \leftarrow k+1$
6: **end while**
7: $q \leftarrow \begin{cases} k & \text{if } \Psi_k(t) \not\equiv 0 \\ k-1 & \text{if } \Psi_k(t) \equiv 0 \end{cases}$

Output: $\{\Psi_k(t)\}_{k=0}^{q}$

Clearly, Algorithm 1 (Sturm-$\mathcal{R}$) is a modification of the Euclidean algorithm for computing $\mathrm{GCD}(\Psi_0(t), \Psi_1(t))$, which generates the *standard Sturm sequence* for a pair of polynomials $\Psi_0(t)$ and $\Psi_1(t)$ in the sense of Henrici (1974, Section 6.3.III), Marden (1989, Section 38) and Mishra (1993, Definition 8.4.2). The Sturm sequence $\{\Psi_k(t)\}_{k=0}^{q}$ and the *polynomial remainder sequence* $\{P_k(t)\}_{k=0}^{q}$ generated by the Euclidean algorithm have the following close relation.

Remark 3 (Relation between the standard Sturm sequence and the polynomial remainder sequence) The Euclidean algorithm for computing $\mathrm{GCD}(\Psi_0, \Psi_1)$ generates the *polynomial remainder sequence* $\{P_k(t)\}_{k=0}^{q}$, where $P_0(t) := \Psi_0(t) := \frac{A_{(0)}(t)}{(t-a)^{e_0}}$, $P_1(t) := \Psi_1(t) := \frac{A_{(1)}(t)}{(t-a)^{e_1}}$ and $P_{k+1}(t)$ $(k = 1, 2, \ldots, q-1)$ are defined inductively by

$$P_{k+1}(t) := P_{k-1}(t) - Q_k(t)P_k(t) \text{ with } Q_k(t) \in \mathbb{R}[t] \text{ and } \deg(P_{k+1}) < \deg(P_k).$$

On the other hand, in Algorithm 1 (Sturm-$\mathcal{R}$), $\Psi_{k+1}(t)$ $(k = 1, 2, \ldots, q-1)$ are defined inductively by

$$\Psi_{k+1}(t) := -\Psi_{k-1}(t) - H_k(t)\Psi_k(t) \text{ with } H_k(t) \in \mathbb{R}[t] \text{ and } \deg(\Psi_{k+1}) < \deg(\Psi_k).$$

As a result, we have

$$\Psi_k(t) = (-1)^{\frac{(k-1)k}{2}} P_k(t) \quad (k = 0, 1, \ldots, q). \tag{8}$$

The following lemma presents useful properties of $\{\Psi_k(t)\}_{k=0}^q$. The properties (a), (b) and (c) are well-known (see, e.g., Henrici 1974, Theorem 6.3b, Marden 1989, Section 38). The properties (d) and (e) will be used in the proposed algebraic phase unwrapping (Theorem 1).

Lemma 1 (Properties of the Sturm sequence generated by Algorithm 1) *The Sturm sequence $\{\Psi_k(t)\}_{k=0}^q$ generated by Algorithm 1 (Sturm-$\mathcal{R}$) satisfies the following properties.*

(a) $\Psi_q(t) \neq 0$ *for* $a \leq t \leq b$,

(b) $\Psi_k(t) \neq 0$ *or* $\Psi_{k+1}(t) \neq 0$ *for* $a \leq t \leq b$ $(k = 0, 1, \dots, q-1)$,

(c) $\Psi_k(t^*) = 0$ *at* $t^* \in [a, b] \Rightarrow \Psi_{k-1}(t^*)\Psi_{k+1}(t^*) < 0$ $(k = 1, 2, \dots, q-1)$,

(d) $\mathrm{sgn}(A_{(0)}(t)) = \mathrm{sgn}(\Psi_0(t))$ *and* $\mathrm{sgn}(A_{(1)}(t)) = \mathrm{sgn}(\Psi_1(t))$ *for* $a < t \leq b$,

(e) $\lim_{t \to a+0} \mathrm{sgn}(A_{(0)}(t)) = \mathrm{sgn}(\Psi_0(a)) \neq 0$ *and* $\lim_{t \to a+0} \mathrm{sgn}(A_{(1)}(t)) = \mathrm{sgn}(\Psi_1(a)) \neq 0$.

Proof See Appendix 3.

The next theorem presents an exact expression of $\theta_A(t^*)$ in Problem 2. The expression (9) does not require any root finding or any numerical integration technique. This theorem is a relaxation of Yamada and Oguchi (2011, Theorem 1). Indeed, Theorem 1 can deal with a special case $A_{(0)}(a) = 0$ which is excluded in Yamada and Oguchi (2011, Theorem 1).

Theorem 1 (Algebraic phase unwrapping for a univariate complex polynomial along the real axis) *Let $\{\Psi_k(t)\}_{k=0}^q$ be the sequence of real polynomials generated by applying Algorithm 1 (Sturm-$\mathcal{R}$) to $A_{(0)}(t), A_{(1)}(t) \in \mathbb{R}[t]$ under the assumptions $A(t) := A_{(0)}(t) + jA_{(1)}(t) \neq 0$ $(t \in [a, b])$, $A_{(0)}(t) \not\equiv 0$ and $A_{(1)}(t) \not\equiv 0$. Define at each $t \in [a, b]$ the number of variations in the sign of $\{\Psi_k(t)\}_{k=0}^q$ by*

$$\begin{aligned} V\{\Psi(t)\} &:= V\{\Psi_0(t), \Psi_1(t), \dots, \Psi_q(t)\} \\ &:= \left|\{i \mid 0 \leq i < q \text{ and } \Psi_i(t)\Psi_{i+\varrho(i)}(t) < 0\}\right|, \end{aligned}$$

where $\varrho(i) := \min\{k \in \mathbb{N}^* \mid \Psi_{i+k}(t) \neq 0\}$. Then, for every $t^* \in (a, b]$, we have

$$\begin{aligned} \theta_A(t^*) = \theta_A(a) &- \begin{Bmatrix} \arctan\{\mathcal{Q}_A(a)\} & \text{if } A_{(0)}(a) \neq 0, \\ \mathrm{sgn}(\Psi_0(a)\Psi_1(a))\pi/2 & \text{if } A_{(0)}(a) = 0, \end{Bmatrix} \\ &+ \begin{cases} \arctan\{\mathcal{Q}_A(t^*)\} + [V\{\Psi(t^*)\} - V\{\Psi(a)\}]\pi & \text{if } A_{(0)}(t^*) \neq 0, \\ \pi/2 + [V\{\Psi(t^*)\} - V\{\Psi(a)\}]\pi & \text{if } A_{(0)}(t^*) = 0. \end{cases} \end{aligned} \tag{9}$$

Proof See Appendix 4.

Example 1 (Expression of the exact unwrapped phase by Theorem 1) Let us construct the unwrapped phase $\theta_A(t)$ $(0 \leq t \leq 1)$ of the univariate complex polynomial

$$\begin{aligned} A(t) &:= A_{(0)}(t) + jA_{(1)}(t) \\ &= (t^4 - 1.11t^3 + 0.356t^2 - 0.0255t) \\ &\quad + j(t^4 - 2.525t^3 + 2.29995t^2 - 0.906172t + 0.131222) \end{aligned}$$

without using any root finding or any numerical intergartion.

Applying Algorithm 1 to $A_{(0)}(t)$ and $A_{(1)}(t)$ for $a=0$ and $b=1$, we obtain the Sturm sequence $\{\Psi_k(t)\}_{k=0}^{5}$ as

$$\begin{aligned}
\Psi_0(t) &= t^3 - \tfrac{111}{100}t^2 + \tfrac{89}{250}t - \tfrac{51}{2000},\\
\Psi_1(t) &= t^4 - \tfrac{101}{40}t^3 + \tfrac{45999}{20000}t^2 - \tfrac{226543}{250000}t + \tfrac{65611}{500000},\\
\Psi_2(t) &= -t^3 + \tfrac{111}{100}t^2 - \tfrac{89}{250}t + \tfrac{51}{2000},\\
\Psi_3(t) &= -\tfrac{3733}{10000}t^2 + \tfrac{94233}{250000}t - \tfrac{190279}{2000000},\\
\Psi_4(t) &= -\tfrac{27788829033}{260102169185000}t + \tfrac{15335859}{278705780000},\\
\Psi_5(t) &= \tfrac{339145264784010639558466646077921 1811}{11996717727057501597535406952577469520000 0}.
\end{aligned}$$

From $A_{(0)}(0) = 0$ and $A_{(1)}(0) = \frac{65611}{500000}$, we have $\theta_A(0) = \pi/2$. Moreover, from $\mathrm{sgn}(\Psi_0(0)\Psi_1(0)) = \mathrm{sgn}\left(-\frac{3346161}{1000000000}\right) = -1$ and $V\{\Psi(0)\} = V\left\{-\frac{51}{2000}, \frac{65611}{500000}, \frac{51}{2000}, -\frac{190279}{2000000}, \frac{15335859}{278705780000}, \frac{3391452647840106395584666460779211811}{119967177270575015975354069525774695200000}\right\} = 3$, the unwrapped phase $\theta_A(t)$ in (9) is expressed as

$$\theta_A(t) = \pi + \begin{cases} \arctan\{\mathcal{Q}_A(t)\} + [V\{\Psi(t)\} - 3]\pi & \text{if } A_{(0)}(t) \neq 0,\\ \pi/2 + [V\{\Psi(t)\} - 3]\pi & \text{if } A_{(0)}(t) = 0, \end{cases}$$

which is depicted in Fig. 1. The correctness of the above result is confirmed by applying the alternative expression (7) if the following information is available.

$$A(t) := t(t-0.1)(t-0.5)(t-0.51) + j(t-0.49)(t-0.515)(t-0.52)(t-1).$$

From Fig. 1, we observe that the unwrapped phase function θ_A can vary rapidly even if $\deg(A)$ is small, which suggests the inherent difficulty in phase unwrapping problem. Moreover, we also observe that the necessary number of digits to express the coefficients of $\{\Psi_k(t)\}_{k=0}^{q}$ grows quickly. This phenomenon is called the *coefficient growth*, which causes numerical instabilities in the direct computer implementation of Algorithm 1 (Sturm-$\mathcal{R}$) (see Section 4.1).

Theorem 1 can also be applied to the computation of the unwrapped phase for bivariate polynomials. Although the two-dimensional phase unwrapping will be discussed much more in detail in Section 3.3, we present here a straightforward application of Theorem 1 to the two-dimensional phase unwrapping.

Example 2 (Phase unwrapping for a bivariate complex polynomial along γ) Let us construct the unwrapped phase, along the path $\gamma(t) := (t, 2t+1)$ $(0 \le t \le 1)$, of the bivariate polynomial

$$f(x,y) := (x^2y^3 - xy^2 - x^2 - 9x - 5y + 16) + j(x^4y + x^2y^3 - 3x^4 + 10xy - 25x - 3).$$

In this case, $F(t) := f(\gamma(t)) = F_{(0)}(t) + jF_{(1)}(t)$ is given by

$$\begin{aligned}
F_{(0)}(t) &= t^2(2t+1)^3 - t(2t+1)^2 - t^2 - 9t - 5(2t+1) + 16\\
&= 8t^5 + 12t^4 + 2t^3 - 4t^2 - 20t + 11,\\
F_{(1)}(t) &= t^4(2t+1) + t^2(2t+1)^3 - 3t^4 + 10t(2t+1) - 25t - 3\\
&= 10t^5 + 10t^4 + 6t^3 + 21t^2 - 15t - 3.
\end{aligned}$$

Applying Algorithm 1 to $F_{(0)}(t)$ and $F_{(1)}(t)$, we can compute the unwrapped phase $\theta_F(t)$. Figure 2a depicts $\theta_F(t)$, and 2b depicts θ_f along γ on the x-y plane. From Fig. 2, we observe

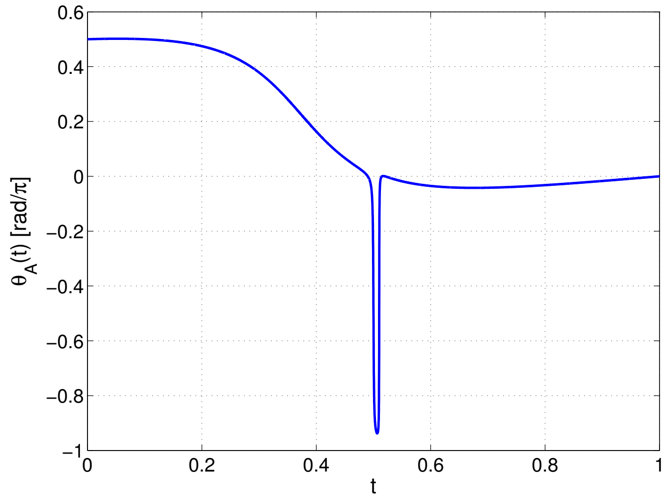
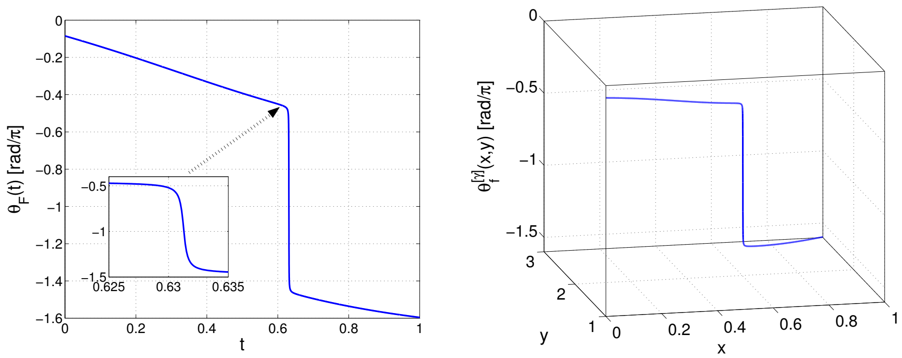


Fig. 1 Exact unwrapped phase by Theorem 1



(a) Exact unwrapped phase along t-axis

(b) Exact unwrapped phase on x-y plane

Fig. 2 Phase unwrapping for a bivariate complex polynomial along γ

that the two-dimennsional unwrapped phase can vary rapidly even for bivariate polynomials of low degrees. Obviously, this notable feature is hardly detectable by most exiting phase unwrapping algorithms, i.e., Busbee et al. (1970), Goldstein et al. (1988), Judge and Bryanston-Cross (1994), Lin et al. (1994), Pritt and Shipman (1994), Buckland et al. (1995), Ghiglia and Romero (1996), Flynn (1997), Costantini (1998), Ying (2006), essentially based on discrete approximations.

3.2 Extension of the algebraic phase unwrapping for a pair of piecewise polynomials

The unwrapped phase of a pair of piecewise polynomials $(S_{(0)}, S_{(1)})$ can be computed by using Theorem 1 repeatedly in each subinterval, which is divided at the knots of the piecewise polynomials. (Note: In the next proposition, for simplicity, we assume $A_{(0)}^{\langle l \rangle}(t) \not\equiv 0$ and

$A_{(1)}^{\langle l\rangle}(t) \not\equiv 0$ in each subinterval $[\xi_l, \xi_{l+1}]$. However, even if this condition is violated, we can compute the unwrapped phase $\theta_S(t)$ by using Proposition 1).

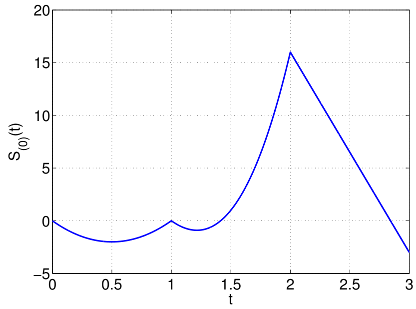
Proposition 2 (Algebraic phase unwrapping for a pair of continuous piecewise polynomials) *Let $S_{(i)} : [\xi_0, \xi_n] \to \mathbb{R}$ $(i = 0, 1)$ be a pair of continuous piecewise polynomials with knots $\xi_l \in (\xi_0, \xi_n)$ $(\xi_1 < \xi_2 < \cdots < \xi_{n-1})$, i.e., $S_{(i)}(t) = A_{(i)}^{\langle l\rangle}(t) \in \mathbb{R}[t]$ in each subinterval $[\xi_l, \xi_{l+1}]$ and $A_{(i)}^{\langle l\rangle}(\xi_{l+1}) = A_{(i)}^{\langle l+1\rangle}(\xi_{l+1})$ $(i = 0, 1$ and $l = 0, 1, \ldots, n-2)$, satisfying $S(t) := S_{(0)}(t) + jS_{(1)}(t) \neq 0$, $A_{(0)}^{\langle l\rangle}(t) \not\equiv 0$ and $A_{(1)}^{\langle l\rangle}(t) \not\equiv 0$ in each subinterval $[\xi_l, \xi_{l+1}]$. Define the sequence of real polynomials $\{\Psi_k^{\langle l\rangle}(t)\}_{k=0}^{q_{\langle l\rangle}}$ $(l = 0, 1, \ldots, n-1)$ by applying Algorithm 1 to $A_{(0)}^{\langle l\rangle}(t)$ and $A_{(1)}^{\langle l\rangle}(t)$ for $a := \xi_l$ and $b := \xi_{l+1}$. Then, for each $t^* \in (\xi_l, \xi_{l+1}]$ $(l = 0, 1, \ldots, n-1)$ we have*

$$
\begin{aligned}
\theta_S(t^*) &= \theta_S(\xi_0) + \int_{\xi_0}^{t^*} \frac{S'_{(1)}(t)S_{(0)}(t) - S_{(1)}(t)S'_{(0)}(t)}{\{S_{(0)}(t)\}^2 + \{S_{(1)}(t)\}^2} dt \\
&= \theta_S(\xi_l) + \int_{\xi_l}^{t^*} \frac{(A_{(1)}^{\langle l\rangle}(t))' A_{(0)}^{\langle l\rangle}(t) - A_{(1)}^{\langle l\rangle}(t)(A_{(0)}^{\langle l\rangle}(t))'}{\{A_{(0)}^{\langle l\rangle}(t)\}^2 + \{A_{(1)}^{\langle l\rangle}(t)\}^2} dt \\
&= \theta_S(\xi_l) - \left\{ \begin{array}{ll} \arctan\{\mathcal{Q}_A^{\langle l\rangle}(\xi_l)\} & \text{if } A_{(0)}^{\langle l\rangle}(\xi_l) \neq 0, \\ \operatorname{sgn}(\Psi_0^{\langle l\rangle}(\xi_l)\Psi_1^{\langle l\rangle}(\xi_l))\pi/2 & \text{if } A_{(0)}^{\langle l\rangle}(\xi_l) = 0, \end{array} \right\} \\
&\quad + \left\{ \begin{array}{ll} \arctan\{\mathcal{Q}_A^{\langle l\rangle}(t^*)\} + [V\{\Psi^{\langle l\rangle}(t^*)\} - V\{\Psi^{\langle l\rangle}(\xi_l)\}]\pi & \text{if } A_{(0)}^{\langle l\rangle}(t^*) \neq 0, \\ \pi/2 + [V\{\Psi^{\langle l\rangle}(t^*)\} - V\{\Psi^{\langle l\rangle}(\xi_l)\}]\pi & \text{if } A_{(0)}^{\langle l\rangle}(t^*) = 0, \end{array} \right.
\end{aligned}
$$

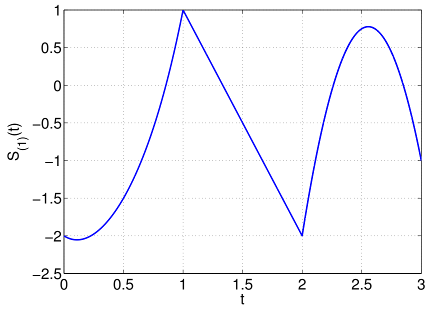
where (i) $\mathcal{Q}_A^{\langle l\rangle}(t) := \dfrac{A_{(1)}^{\langle l\rangle}(t)}{A_{(0)}^{\langle l\rangle}(t)}$ $(l = 0, 1, \ldots, n-1)$, (ii) $\theta_S(\xi_0) \in (-\pi, \pi]$ satisfies $S(\xi_0) = |S(\xi_0)|e^{j\theta_S(\xi_0)}$, and (iii) $\theta_S(\xi_l)$ is given by

$$
\begin{aligned}
\theta_S(\xi_l) &= \theta_S(\xi_0) + \sum_{i=0}^{l-1} \int_{\xi_i}^{\xi_{i+1}} \frac{(A_{(1)}^{\langle i\rangle}(t))' A_{(0)}^{\langle i\rangle}(t) - A_{(1)}^{\langle i\rangle}(t)(A_{(0)}^{\langle i\rangle}(t))'}{\{A_{(0)}^{\langle i\rangle}(t)\}^2 + \{A_{(1)}^{\langle i\rangle}(t)\}^2} dt \\
&= \theta_S(\xi_0) + \sum_{i=0}^{l-1} \left[- \left\{ \begin{array}{ll} \arctan\{\mathcal{Q}_A^{\langle i\rangle}(\xi_i)\} & \text{if } A_{(0)}^{\langle i\rangle}(\xi_i) \neq 0, \\ \operatorname{sgn}(\Psi_0^{\langle i\rangle}(\xi_i)\Psi_1^{\langle i\rangle}(\xi_i))\pi/2 & \text{if } A_{(0)}^{\langle i\rangle}(\xi_i) = 0, \end{array} \right\} \right. \\
&\quad \left. + \left\{ \begin{array}{ll} \arctan\{\mathcal{Q}_A^{\langle i\rangle}(\xi_{i+1})\} + [V\{\Psi^{\langle i\rangle}(\xi_{i+1})\} - V\{\Psi^{\langle i\rangle}(\xi_i)\}]\pi & \text{if } A_{(0)}^{\langle i\rangle}(\xi_{i+1}) \neq 0, \\ \pi/2 + [V\{\Psi^{\langle i\rangle}(\xi_{i+1})\} - V\{\Psi^{\langle i\rangle}(\xi_i)\}]\pi & \text{if } A_{(0)}^{\langle i\rangle}(\xi_{i+1}) = 0 \end{array} \right. \right].
\end{aligned}
$$

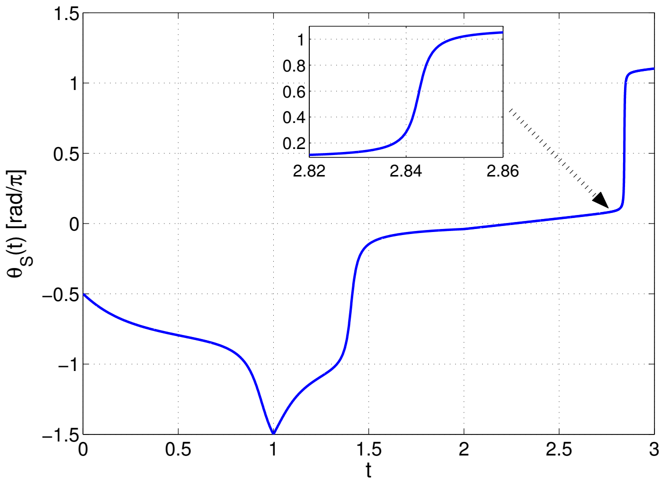
Proof The proof is obvious from Theorem 1 and the definition of $(S_{(0)}, S_{(1)})$.



(a) Piecewise polynomial $S_{(0)}(t)$



(b) Piecewise polynomial $S_{(1)}(t)$



(c) Exact unwrapped phase by Proposition 2

Fig. 3 Phase unwrapping for a pair of piecewise polynomials

Example 3 (Phase unwrapping for a pair of piecewise polynomials) Let us construct the unwrapped phase of the function $S(t) := S_{(0)}(t) + jS_{(1)}(t)$, where

$$S_{(0)}(t) := \begin{cases} 8t^2 - 8t & \text{if } 0 \leq t \leq 1, \\ 8t^3 - 8t^2 - 16t + 16 & \text{if } 1 \leq t \leq 2, \\ -19t + 54 & \text{if } 2 \leq t \leq 3, \end{cases}$$

$$S_{(1)}(t) := \begin{cases} 2t^4 - 3t^3 + 5t^2 - t - 2 & \text{if } 0 \leq t \leq 1, \\ -3t + 4 & \text{if } 1 \leq t \leq 2, \\ -9t^2 + 46t - 58 & \text{if } 2 \leq t \leq 3. \end{cases}$$

Figure 3a, b depict $S_{(0)}(t)$ and $S_{(1)}(t)$ respectively. From Proposition 2, by applying the algebraic phase unwrapping (Theorem 1) repeatedly in each subinterval $[0,1]$, $[1,2]$ and $[2,3]$, we can compute the unwrapped phase $\theta_S(t)$ for $t \in [0,3]$. Figure 3c depicts the obtained unwrapped phase $\theta_S(t)$.

3.3 Extension of the algebraic phase unwrapping to a pair of bivariate polynomials

In this section, we consider the two-dimensional phase unwrapping problem. Unlike one-dimensional cases in Sects. 3.1 and 3.2, the continuity of the unwrapped phase can not necessarily be guaranteed globally in $\mathbb{R}^2$. Let us start with such a simplest example.

Example 4 (Path dependence of two-dimensional unwrapped phase I) We consider the phase unwrapping for $f(x,y) := f_{(0)}(x,y) + jf_{(1)}(x,y) := x + jy$ along different piecewise C^1 paths

$$\gamma^{\mathrm{I}}_{(x^*,y^*)}(t) := \begin{cases} (t-1,-1) & \text{if } 0 \le t \le x^*+1, \\ (x^*, t-x^*-2) & \text{if } x^*+1 \le t \le x^*+y^*+2, \end{cases}$$

and

$$\gamma^{\mathrm{II}}_{(x^*,y^*)}(t) := \begin{cases} (-1,t-1) & \text{if } 0 \le t \le y^*+1, \\ (t-y^*-2, y^*) & \text{if } y^*+1 \le t \le x^*+y^*+2. \end{cases}$$

in $[-1,1]\times[-1,1] \subset \mathbb{R}^2$. Both paths connect $(-1,-1) \in \mathbb{R}^2$ and $(x^*,y^*) \in \mathbb{R}^2$ in $[-1,1]\times[-1,1] \subset \mathbb{R}^2$. By the definition, for each $(x^*,y^*) \in [-1,1]\times[-1,1]$, the unwrapped phase $\theta_f^{[\gamma^K]}(x^*,y^*)$ for f along $\gamma^K_{(x^*,y^*)}$ $(K = \mathrm{I}, \mathrm{II})$ are expressed respectively by

$$\begin{aligned} \theta_f^{[\gamma^{\mathrm{I}}]}(x^*,y^*) &:= \theta_f^{[\gamma^{\mathrm{I}}]}(-1,-1) + \int_0^{x^*+y^*+2} \Im\left\{ \frac{\left(f_{(0)}(\gamma^{\mathrm{I}}_{(x^*,y^*)}(t))\right)' + j\left(f_{(1)}(\gamma^{\mathrm{I}}_{(x^*,y^*)}(t))\right)'}{f_{(0)}(\gamma^{\mathrm{I}}_{(x^*,y^*)}(t)) + jf_{(1)}(\gamma^{\mathrm{I}}_{(x^*,y^*)}(t))} \right\} dt \\ &= -\frac{3\pi}{4} + \int_0^{x^*+1} \Im\left\{ \frac{(t-1)'}{t-1-j} \right\} dt + \int_{x^*+1}^{x^*+y^*+2} \Im\left\{ \frac{j(t-x^*-2)'}{x^*+j(t-x^*-2)} \right\} dt, \end{aligned}$$

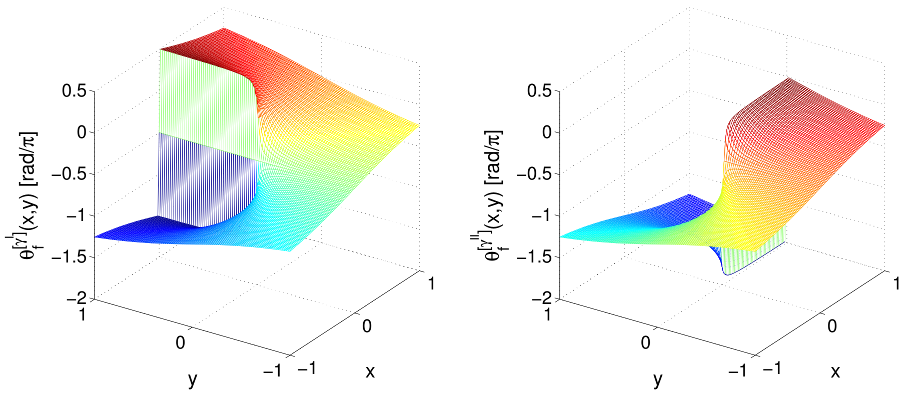
and

$$\begin{aligned} \theta_f^{[\gamma^{\mathrm{II}}]}(x^*,y^*) &:= \theta_f^{[\gamma^{\mathrm{II}}]}(-1,-1) + \int_0^{x^*+y^*+2} \Im\left\{ \frac{\left(f_{(0)}(\gamma^{\mathrm{II}}_{(x^*,y^*)}(t))\right)' + j\left(f_{(1)}(\gamma^{\mathrm{II}}_{(x^*,y^*)}(t))\right)'}{f_{(0)}(\gamma^{\mathrm{II}}_{(x^*,y^*)}(t)) + jf_{(1)}(\gamma^{\mathrm{II}}_{(x^*,y^*)}(t))} \right\} dt \\ &= -\frac{3\pi}{4} + \int_0^{y^*+1} \Im\left\{ \frac{j(t-1)'}{-1+j(t-1)} \right\} dt + \int_{y^*+1}^{x^*+y^*+2} \Im\left\{ \frac{(t-y^*-2)'}{t-y^*-2+jy^*} \right\} dt. \end{aligned}$$

The unwrapped phases $\theta_f^{[\gamma^K]}(x^*,y^*)$ $(K = \mathrm{I}, \mathrm{II})$ for $(x^*,y^*) \in [-1,1]\times[-1,1]$ can be computed, through the equivalent expression in (3), by Theorem 1 or by Proposition 2. The results are depicted in Fig. 4a, b.

From Fig. 4, we observe that both unwrapped phases $\theta_f^{[\gamma^{\mathrm{I}}]}(x,y)$ and $\theta_f^{[\gamma^{\mathrm{II}}]}(x,y)$ are continuous along the paths $\gamma^{\mathrm{I}}_{(x^*,y^*)}$ and $\gamma^{\mathrm{II}}_{(x^*,y^*)}$ respectively, but are not continuous as real valued functions defined over $[-1,1]\times[-1,1]$. Moreover, the values of $\theta_f^{[\gamma^{\mathrm{I}}]}(x,y)$ and $\theta_f^{[\gamma^{\mathrm{II}}]}(x,y)$ behave very differently over $[0,1]\times[0,1]$. This example implies that the two-dimensional unwrapped phase generally depends on the path along which it is defined. (Note: In Fig. 4a, the unwrapped phase $\theta_f^{[\gamma^{\mathrm{I}}]}(x,y) = -\pi/2$ for $x = 0$ based on the definition (4). Hence, the unwrapped phase $\theta_f^{[\gamma^{\mathrm{I}}]}$ does not guarantee $f(x,y) = |f(x,y)|e^{j\theta_f^{[\gamma^{\mathrm{I}}]}(x,y)}$ for $(x,y) \in \{0\}\times(0,1]$. Similarly, in Fig. 4b, $\theta_f^{[\gamma^{\mathrm{II}}]}$ does not guarantee $f(x,y) = |f(x,y)|e^{j\theta_f^{[\gamma^{\mathrm{II}}]}(x,y)}$ for $(x,y) \in (0,1]\times\{0\}$).

Next, we revisit the bivariate polynomial defined in Example 2, and observe the path dependence of the two-dimensional phase unwrapping again.



(a) The unwrapped phase θ_f along $\gamma^{\mathrm{I}}_{(x^*,y^*)}$ (b) The unwrapped phase θ_f along $\gamma^{\mathrm{II}}_{(x^*,y^*)}$

Fig. 4 Two different unwrapped phases for $f(x,y) = x + jy$

Example 5 (Path dependence of two-dimensional unwrapped phase II) We revisit the bivariate polynomial, defined in Example 2,

$$f(x,y) := (x^2y^3 - xy^2 - x^2 - 9x - 5y + 16) + j(x^4y + x^2y^3 - 3x^4 + 10xy - 25x - 3).$$

This polynomial has a zero at $(x,y) = (0.642303812449619, 2.252655013605015)$. For any $(x^*,y^*) \in [0,\infty) \times [0,\infty)$, we define two piecewise C^1 paths $\gamma^{\mathrm{I}}_{(x^*,y^*)} : [0, x^*+y^*] \to \mathbb{R}^2$ and $\gamma^{\mathrm{II}}_{(x^*,y^*)} : [0, x^*+y^*] \to \mathbb{R}^2$ as

$$\gamma^{\mathrm{I}}_{(x^*,y^*)}(t) := \begin{cases} (t,0) & \text{if } 0 \le t \le x^*, \\ (x^*, t - x^*) & \text{if } x^* \le t \le x^* + y^*, \end{cases}$$

and

$$\gamma^{\mathrm{II}}_{(x^*,y^*)}(t) := \begin{cases} (0,t) & \text{if } 0 \le t \le y^*, \\ (t - y^*, y^*) & \text{if } y^* \le t \le x^* + y^*. \end{cases}$$

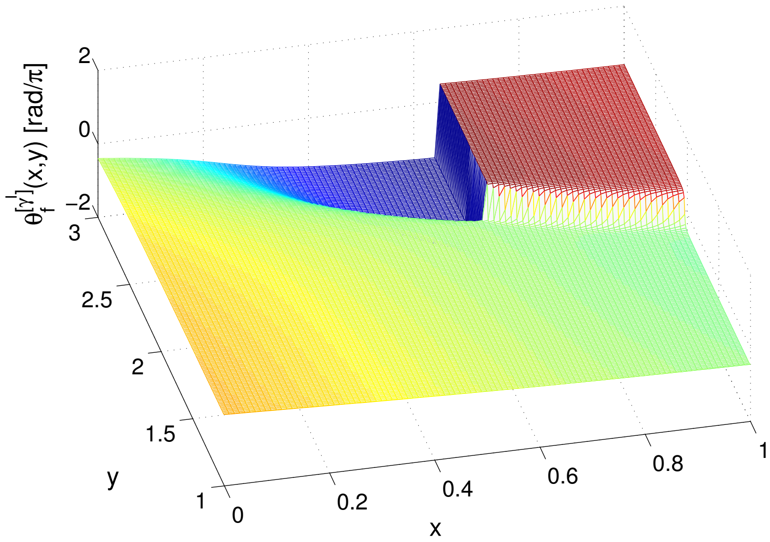
The paths $\gamma^{K}_{(x^*,y^*)}$ $(K = \mathrm{I}, \mathrm{II})$ have same initial points $\gamma^{\mathrm{I}}_{(x^*,y^*)}(0) = \gamma^{\mathrm{II}}_{(x^*,y^*)}(0) = (0,0)$ and final points are $\gamma^{\mathrm{I}}_{(x^*,y^*)}(x^*+y^*) = \gamma^{\mathrm{II}}_{(x^*,y^*)}(x^*+y^*) = (x^*,y^*)$. By defining ${}^{K}F_{(i)}(t) \in \mathbb{R}[t]$ and ${}^{K}F(t) \in \mathbb{R}[t] + j\mathbb{R}[t]$ $(K = \mathrm{I}, \mathrm{II}$ and $i = 0,1)$ as

$${}^{K}F_{(i)}(t) := f_{(i)}(\gamma^{K}_{(x^*,y^*)}(t)) \text{ and } {}^{K}F(t) := {}^{K}F_{(0)}(t) + j\,{}^{K}F_{(1)}(t).$$

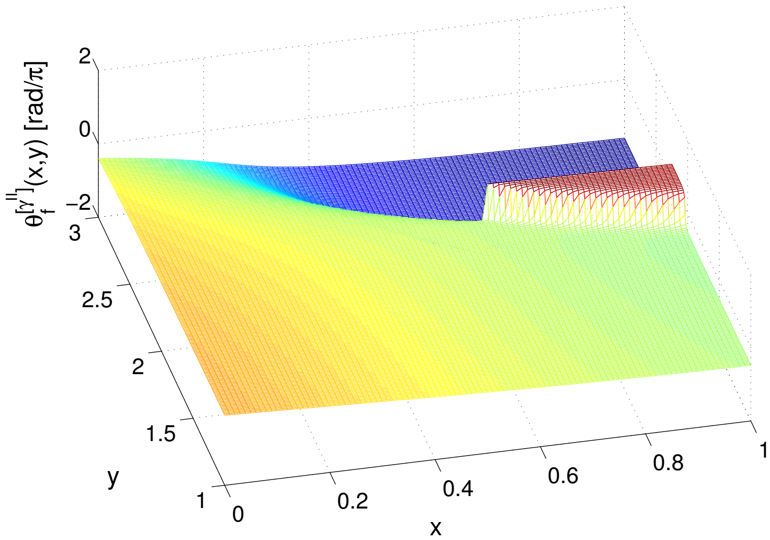
Let us compute, for $K = \mathrm{I}, \mathrm{II}$,

$$\theta_f^{[\gamma^K]}(x^*,y^*) := \theta_{{}^{K}F}(x^*+y^*) := \theta_{{}^{K}F}(0) + \int_0^{x^*+y^*} \Im\left\{ \frac{{}^{K}F'_{(0)}(t) + j\,{}^{K}F'_{(1)}(t)}{{}^{K}F_{(0)}(t) + j\,{}^{K}F_{(1)}(t)} \right\} dt$$

by using Proposition 2. Figure 5a, b depict $\theta_f^{[\gamma^{\mathrm{I}}]}$ and $\theta_f^{[\gamma^{\mathrm{II}}]}$ respectively.



(a) The unwrapped phase θ_f along $\gamma^{\mathrm{I}}_{(x^*,y^*)}$



(b) The unwrapped phase θ_f along $\gamma^{\mathrm{II}}_{(x^*,y^*)}$

Fig. 5 Two different unwrapped phases for the bivariate complex polynomial defined in Example 2

From Examples 4 and 5, we verify that the unwrapped phase θ_f in general depends on the path of integral.

The next theorem presents a condition which guarantees (i) the unique existence of the two-dimensional unwrapped phase as a C^2 function and (ii) the path independence of the unwrapped phase.

Theorem 2 (Path independence of two-dimensional phase unwrapping) *Let $D \subset \mathbb{R}^2$ be a simply connected domain. Suppose that $f_{(i)} : \mathbb{R}^2 \to \mathbb{R}$ $(i = 0, 1)$ are $C^2(D)$ functions satisfying $f(x,y) := f_{(0)}(x,y) + jf_{(1)}(x,y) \neq 0$ for all $(x,y) \in D$. Then the following hold.*

(a) (A symmetry of second derivatives)

$$\frac{\partial}{\partial x}\left[\Im\left\{\frac{\frac{\partial f_{(0)}}{\partial y}(x,y) + j\frac{\partial f_{(1)}}{\partial y}(x,y)}{f_{(0)}(x,y) + jf_{(1)}(x,y)}\right\}\right] = \frac{\partial}{\partial y}\left[\Im\left\{\frac{\frac{\partial f_{(0)}}{\partial x}(x,y) + j\frac{\partial f_{(1)}}{\partial x}(x,y)}{f_{(0)}(x,y) + jf_{(1)}(x,y)}\right\}\right]$$

for all $(x,y) \in D$.

(b) (Unique existence of two-dimensional unwrapped phase) *Suppose that $\theta_0 \in (-\pi, \pi]$ satisfying $f(x_0,y_0) = |f(x_0,y_0)|e^{j\theta_0}$ is given at some $(x_0,y_0) \in D$, then there exist a unique function $\theta_f \in C^2(D)$ satisfying*

$$\theta_f(x_0,y_0) = \theta_0,$$

$$\frac{\partial \theta_f}{\partial x}(x,y) = \Im\left\{\frac{\frac{\partial f_{(0)}}{\partial x}(x,y) + j\frac{\partial f_{(1)}}{\partial x}(x,y)}{f_{(0)}(x,y) + jf_{(1)}(x,y)}\right\} \text{ for all } (x,y) \in D,$$

and

$$\frac{\partial \theta_f}{\partial y}(x,y) = \Im\left\{\frac{\frac{\partial f_{(0)}}{\partial y}(x,y) + j\frac{\partial f_{(1)}}{\partial y}(x,y)}{f_{(0)}(x,y) + jf_{(1)}(x,y)}\right\} \text{ for all } (x,y) \in D.$$

θ_f is the scalar potential of the vector field

$$\left(\Im\left\{\frac{\frac{\partial f_{(0)}}{\partial x}(x,y) + j\frac{\partial f_{(1)}}{\partial x}(x,y)}{f_{(0)}(x,y) + jf_{(1)}(x,y)}\right\}, \Im\left\{\frac{\frac{\partial f_{(0)}}{\partial y}(x,y) + j\frac{\partial f_{(1)}}{\partial y}(x,y)}{f_{(0)}(x,y) + jf_{(1)}(x,y)}\right\}\right) \text{ over } D.$$

(c) (Path independence of two-dimensional unwrapped phase I) *Suppose that Ω is a closed subset of D, with positively oriented boundary $\partial\Omega$. Then we have*

$$\oint_{\partial\Omega}\left[\Im\left\{\frac{\frac{\partial f_{(0)}}{\partial x}(x,y) + j\frac{\partial f_{(1)}}{\partial x}(x,y)}{f_{(0)}(x,y) + jf_{(1)}(x,y)}\right\}dx + \Im\left\{\frac{\frac{\partial f_{(0)}}{\partial y}(x,y) + j\frac{\partial f_{(1)}}{\partial y}(x,y)}{f_{(0)}(x,y) + jf_{(1)}(x,y)}\right\}dy\right]$$
$$= \oint_{\partial\Omega}\left[\frac{\partial \theta_f}{\partial x}(x,y)dx + \frac{\partial \theta_f}{\partial y}(x,y)dy\right] = 0.$$

In particular, if γ^{I} and γ^{II} are piecewise C^1 paths in D with the same initial and final points, we have

$$\int_{\gamma^{\mathrm{I}}}\left[\frac{\partial \theta_f}{\partial x}(x,y)dx + \frac{\partial \theta_f}{\partial y}(x,y)dy\right] = \int_{\gamma^{\mathrm{II}}}\left[\frac{\partial \theta_f}{\partial x}(x,y)dx + \frac{\partial \theta_f}{\partial y}(x,y)dy\right].$$

(d) (Path independence of two-dimensional unwrapped phase II) *Suppose γ^{I} and γ^{II} are piecewise C^1 paths in D with the same initial and final points, i.e., $\gamma^{\mathrm{I}} : [a,b] \to D$ and*

$\gamma^{\mathrm{II}} : [c,d] \to D$ satisfy $\gamma^{\mathrm{I}}(a) = \gamma^{\mathrm{II}}(c) = (x_0,y_0) \in D$ and $\gamma^{\mathrm{I}}(b) = \gamma^{\mathrm{II}}(d) = (x_1,y_1) \in D$. Then

$$\begin{aligned}\theta_f(x_1,y_1) &= \theta_f(x_0,y_0) + \int_a^b \Im\left\{\frac{\left(f_{(0)}(\gamma^{\mathrm{I}}(t))\right)' + j\left(f_{(1)}(\gamma^{\mathrm{I}}(t))\right)'}{f_{(0)}(\gamma^{\mathrm{I}}(t)) + jf_{(1)}(\gamma^{\mathrm{I}}(t))}\right\} dt \\ &= \theta_f(x_0,y_0) + \int_c^d \Im\left\{\frac{\left(f_{(0)}(\gamma^{\mathrm{II}}(\tau))\right)' + j\left(f_{(1)}(\gamma^{\mathrm{II}}(\tau))\right)'}{f_{(0)}(\gamma^{\mathrm{II}}(\tau)) + jf_{(1)}(\gamma^{\mathrm{II}}(\tau))}\right\} d\tau.\end{aligned}$$

Proof See Appendix 5.

Theorem 2 guarantees that for any bivariate polynomial $f(x,y) \in \mathbb{R}[x,y] + j\mathbb{R}[x,y]$ not having any zero in a simply connected domain $D \subset \mathbb{R}^2$, the unwrapped phase function $\theta_f \in C^2(D)$ can be defined uniquely with line integrals along any piecewise C^1 path in D. This fact naturally leads to the following algebraic phase unwrapping for $f(x,y) \in \mathbb{R}[x,y] + j\mathbb{R}[x,y]$ over D.

Proposition 3 (Algebraic phase unwrapping for a pair of bivariate polynomials) *Let $f_{(i)}(x,y) \in \mathbb{R}[x,y]$ $(i = 0,1)$ be a pair of bivariate polynomials satisfying $f(x,y) := f_{(0)}(x,y) + jf_{(1)}(x,y) \neq 0$ for all (x,y) in a simply connected domain $D \subset \mathbb{R}^2$. Suppose that $\theta_0 \in (-\pi,\pi]$ satisfying $f(x_0,y_0) = |f(x_0,y_0)|e^{j\theta_0}$ at some $(x_0,y_0) \in D$ is given. Suppose also that two points $(x_0,y_0),(x_n,y_n) \in D$ are connected by a piecewise C^1 path $\gamma : [0,t_n] \to D$ having only horizontal and vertical displacements, satisfying $x_i = x_{i+1}$ or $y_i = y_{i+1}$ at $(x_i,y_i) \in D$ $(i = 0,1,\ldots,n-1)$. More precisely γ is given by*

$$\gamma(t) := \begin{cases} \gamma^{\langle 0\rangle}_{(x_0,y_0)}(t) & \text{for } 0 \le t \le t_1, \\ \gamma^{\langle 1\rangle}_{(x_1,y_1)}(t) & \text{for } t_1 \le t \le t_2, \\ \qquad\vdots & \\ \gamma^{\langle n-1\rangle}_{(x_{n-1},y_{n-1})}(t) & \text{for } t_{n-1} \le t \le t_n, \end{cases}$$

where

$$\gamma^{\langle l\rangle}_{(x_l,y_l)}(t) := \begin{cases} (t - t_l + x_l, y_l) & \text{if } x_{l+1} > x_l \text{ and } y_{l+1} = y_l, \\ (-t + t_l + x_l, y_l) & \text{if } x_{l+1} < x_l \text{ and } y_{l+1} = y_l, \\ (x_l, t - t_l + y_l) & \text{if } y_{l+1} > y_l \text{ and } x_{l+1} = x_l, \\ (x_l, -t + t_l + y_l) & \text{if } y_{l+1} < y_l \text{ and } x_{l+1} = x_l, \end{cases}$$

$t_0 := 0$ and $t_l := \sum_{i=0}^{l-1}(|x_{i+1} - x_i| + |y_{i+1} - y_i|)$ $(l = 1,2,\ldots,n)$. Define the univariate polynomials

$$F_{y_l(i)}(t) := f_{(i)}(t,y_l) \text{ and } F_{x_l(i)}(t) := f_{(i)}(x_l,t) \quad (i = 0,1 \text{ and } l = 0,1,\ldots,n-1),$$

and define the sequences of real polynomials $\{\Psi_{y_lk}(t)\}_{k=0}^{q_{y_l}}$ and $\{\Psi_{x_lk}(t)\}_{k=0}^{q_{x_l}}$ $(l = 0,1,\ldots,n-1)$ by applying Algorithm 1 to nonzero polynomials $(F_{y_l(0)}(t),F_{y_l(1)}(t))$ for $(a,b) = (\min\{x_l,x_{l+1}\},\max\{x_l,x_{l+1}\})$, and $(F_{x_l(0)}(t),F_{x_l(1)}(t))$ for $(a,b) = (\min\{y_l,y_{l+1}\},$

$\max\{y_l, y_{l+1}\}$). *Then the two-dimensional unwrapped phase* θ_f *(in Theorem 2(b)) at* $(x_n, y_n) \in D$ *can be expressed by*

$$\begin{aligned}\theta_f(x_n, y_n) &= \theta_0 + \int_{t_0}^{t_n} \frac{\left(f_{(1)}(\gamma(t))\right)' f_{(0)}(\gamma(t)) - f_{(1)}(\gamma(t))\left(f_{(0)}(\gamma(t))\right)'}{\{f_{(0)}(\gamma(t))\}^2 + \{f_{(1)}(\gamma(t))\}^2} dt \\ &= \theta_0 + \sum_{l=0}^{n-1} \Upsilon(l),\end{aligned}$$

where, for $l = 0, 1, \ldots, n-1$,

$$\begin{aligned}\Upsilon(l) &:= \int_{t_l}^{t_{l+1}} \frac{\left(f_{(1)}(\gamma(t))\right)' f_{(0)}(\gamma(t)) - f_{(1)}(\gamma(t))\left(f_{(0)}(\gamma(t))\right)'}{\{f_{(0)}(\gamma(t))\}^2 + \{f_{(1)}(\gamma(t))\}^2} dt \\
&= \begin{cases} \displaystyle\int_{x_l}^{x_{l+1}} \frac{\left(F_{y_l(1)}(t)\right)' F_{y_l(0)}(t) - F_{y_l(1)}(t)\left(F_{y_l(0)}(t)\right)'}{\{F_{y_l(0)}(t)\}^2 + \{F_{y_l(1)}(t)\}^2} dt & \text{if } x_{l+1} > x_l, \\
\displaystyle -\int_{x_{l+1}}^{x_l} \frac{\left(F_{y_l(1)}(t)\right)' F_{y_l(0)}(t) - F_{y_l(1)}(t)\left(F_{y_l(0)}(t)\right)'}{\{F_{y_l(0)}(t)\}^2 + \{F_{y_l(1)}(t)\}^2} dt & \text{if } x_{l+1} < x_l, \\
\displaystyle\int_{y_l}^{y_{l+1}} \frac{\left(F_{x_l(1)}(t)\right)' F_{x_l(0)}(t) - F_{x_l(1)}(t)\left(F_{x_l(0)}(t)\right)'}{\{F_{x_l(0)}(t)\}^2 + \{F_{x_l(1)}(t)\}^2} dt & \text{if } y_{l+1} > y_l, \\
\displaystyle -\int_{y_{l+1}}^{y_l} \frac{\left(F_{x_l(1)}(t)\right)' F_{x_l(0)}(t) - F_{x_l(1)}(t)\left(F_{x_l(0)}(t)\right)'}{\{F_{x_l(0)}(t)\}^2 + \{F_{x_l(1)}(t)\}^2} dt & \text{if } y_{l+1} < y_l, \end{cases} \\
&= \begin{cases} \left.\begin{array}{l} -\begin{cases} \arctan\{\mathcal{Q}_{F_{y_l}}(x_l)\} & \text{if } F_{y_l(0)}(x_l) \neq 0, \\ \mathrm{sgn}(\Psi_{y_l 0}(x_l)\Psi_{y_l 1}(x_l))\pi/2 & \text{if } F_{y_l(0)}(x_l) = 0, \end{cases} \\ +\begin{cases} \arctan\{\mathcal{Q}_{F_{y_l}}(x_{l+1})\} + [V\{\Psi_{y_l}(x_{l+1})\} - V\{\Psi_{y_l}(x_l)\}]\pi & \text{if } F_{y_l(0)}(x_{l+1}) \neq 0, \\ \pi/2 + [V\{\Psi_{y_l}(x_{l+1})\} - V\{\Psi_{y_l}(x_l)\}]\pi & \text{if } F_{y_l(0)}(x_{l+1}) = 0, \end{cases} \end{array}\right\} & \text{if } x_{l+1} > x_l, \\
\left.\begin{array}{l} +\begin{cases} \arctan\{\mathcal{Q}_{F_{y_l}}(x_{l+1})\} & \text{if } F_{y_l(0)}(x_{l+1}) \neq 0, \\ \mathrm{sgn}(\Psi_{y_l 0}(x_{l+1})\Psi_{y_l 1}(x_{l+1}))\pi/2 & \text{if } F_{y_l(0)}(x_{l+1}) = 0, \end{cases} \\ -\begin{cases} \arctan\{\mathcal{Q}_{F_{y_l}}(x_l)\} + [V\{\Psi_{y_l}(x_l)\} - V\{\Psi_{y_l}(x_{l+1})\}]\pi & \text{if } F_{y_l(0)}(x_l) \neq 0, \\ \pi/2 + [V\{\Psi_{y_l}(x_l)\} - V\{\Psi_{y_l}(x_{l+1})\}]\pi & \text{if } F_{y_l(0)}(x_l) = 0, \end{cases} \end{array}\right\} & \text{if } x_{l+1} < x_l, \\
\left.\begin{array}{l} -\begin{cases} \arctan\{\mathcal{Q}_{F_{x_l}}(y_l)\} & \text{if } F_{x_l(0)}(y_l) \neq 0, \\ \mathrm{sgn}(\Psi_{x_l 0}(y_l)\Psi_{x_l 1}(y_l))\pi/2 & \text{if } F_{x_l(0)}(y_l) = 0, \end{cases} \\ +\begin{cases} \arctan\{\mathcal{Q}_{F_{x_l}}(y_{l+1})\} + [V\{\Psi_{x_l}(y_{l+1})\} - V\{\Psi_{x_l}(y_l)\}]\pi & \text{if } F_{x_l(0)}(y_{l+1}) \neq 0, \\ \pi/2 + [V\{\Psi_{x_l}(y_{l+1})\} - V\{\Psi_{x_l}(y_l)\}]\pi & \text{if } F_{x_l(0)}(y_{l+1}) = 0, \end{cases} \end{array}\right\} & \text{if } y_{l+1} > y_l, \\
\left.\begin{array}{l} +\begin{cases} \arctan\{\mathcal{Q}_{F_{x_l}}(y_{l+1})\} & \text{if } F_{x_l(0)}(y_{l+1}) \neq 0, \\ \mathrm{sgn}(\Psi_{x_l 0}(y_{l+1})\Psi_{x_l 1}(y_{l+1}))\pi/2 & \text{if } F_{x_l(0)}(y_{l+1}) = 0, \end{cases} \\ -\begin{cases} \arctan\{\mathcal{Q}_{F_{x_l}}(y_l)\} + [V\{\Psi_{x_l}(y_l)\} - V\{\Psi_{x_l}(y_{l+1})\}]\pi & \text{if } F_{x_l(0)}(y_l) \neq 0, \\ \pi/2 + [V\{\Psi_{x_l}(y_l)\} - V\{\Psi_{x_l}(y_{l+1})\}]\pi & \text{if } F_{x_l(0)}(y_l) = 0, \end{cases} \end{array}\right\} & \text{if } y_{l+1} < y_l, \end{cases}\end{aligned}$$

$\mathcal{Q}_{F_{x_l}}(t) := \dfrac{F_{x_l(1)}(t)}{F_{x_l(0)}(t)}$ *and* $\mathcal{Q}_{F_{y_l}}(t) := \dfrac{F_{y_l(1)}(t)}{F_{y_l(0)}(t)}$ $(l = 0, 1, \ldots, n-1)$.

Proof The proof is obvious from Theorems 1, 2 and the definitions of γ, F_{y_l} and F_{x_l}.

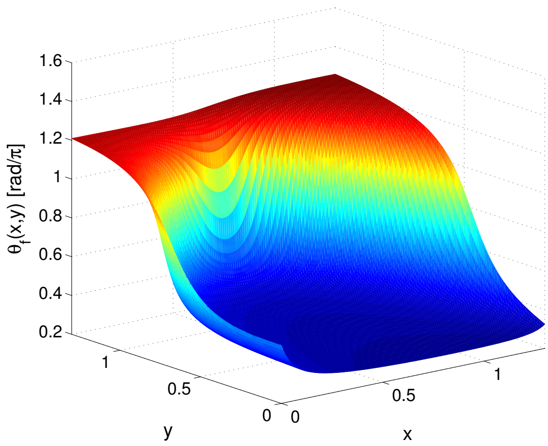


Fig. 6 Exact two-dimensional unwrapped phase by Proposition 3

Example 6 (Phase unwrapping for a pair of bivariate polynomials over $\mathbb{R}^2$) Let us construct the unwrapped phase of the bivariate complex polynomial $f(x,y) := f_{(0)}(x,y) + jf_{(1)}(x,y)$ over $[0,1.3]\times[0,1.3]$ by using Proposition 3, where

$$f_{(0)}(x,y) := x^4y - 4x^4 - 2x^3y - 3xy + 10x - 2y^3,$$
$$f_{(1)}(x,y) := x^4y - 4x^4 - 2x^3y - 3xy + 10x - 2y^3 + 1.$$

Since $f_{(1)}(x,y) = f_{(0)}(x,y) + 1$ for all $(x,y) \in \mathbb{R}^2$, we have $f(x,y) \neq 0$ for all $(x,y) \in \mathbb{R}^2$. For any $(x^*,y^*) \in [0,1.3]\times[0,1.3]$, we choose the piecewise C^1 path $\gamma_{(x^*,y^*)}$ as

$$\gamma_{(x^*,y^*)}(t) := \begin{cases} (t,0) & \text{if } 0 \le t \le x^*, \\ (x^*, t-x^*) & \text{if } x^* \le t \le x^*+y^*, \end{cases}$$

that is $(x_0,y_0) = (0,0)$, $(x_1,y_1) = (x^*,0)$ and $(x_2,y_2) = (x^*,y^*)$.

By applying Algorithm 1 to $F_{y_0(0)}(t) := f_{(0)}(t,0) = -4t^4 + 10t$ and $F_{y_0(1)}(t) := f_{(1)}(t,0) = -4t^4 + 10t + 1$, we obtain the Sturm sequence $\{\Psi_{y_0k}(t)\}_{k=0}^{q_{y_0}}$ as

$$\Psi_{y_00}(t) = -4t^3 + 10,$$
$$\Psi_{y_01}(t) = -4t^4 + 10t + 1,$$
$$\Psi_{y_02}(t) = 4t^3 - 10,$$
$$\Psi_{y_03}(t) = -1.$$

From $F_{y_0(0)}(0) = 0$, $\mathrm{sgn}(\Psi_{y_00}(0)\Psi_{y_01}(0)) = 1$, $V\{\Psi_{y_0}(0)\} = V\{10,1,-10,-1\} = 1$ and $F_{y_0(0)}(x^*) = -4x^{*4} + 10x^* \neq 0$ for all $0 < x^* \le 1.3$, we have

$$\Upsilon_{(x^*,y^*)}(0) = \begin{cases} -\pi/2 + \arctan\left\{\dfrac{-4x^{*4}+10x^*+1}{-4x^{*4}+10x^*}\right\} + [V\{\Psi_{y_0}(x^*)\} - 1]\pi & \text{if } 0 < x^* \le 1.3, \\ 0 & \text{if } x^* = 0. \end{cases}$$

Similarly, for $0 < x^* \le 1.3$, by applying Algorithm 1 to $F_{x_1(0)}(t) := f_{(0)}(x^*,t) = -2t^3 +$

$(x^{*4} - 2x^{*3} - 3x^*)t - 4x^{*4} + 10x^*$ and $F_{x_1(1)}(t) := f_{(1)}(x^*, t) = -2t^3 + (x^{*4} - 2x^{*3} - 3x^*)t - 4x^{*4} + 10x^* + 1$, we obtain the Sturm sequence $\{\Psi_{x_1 k}(t)\}_{k=0}^{q_{x_1}}$ as

$$\begin{aligned}
\Psi_{x_1 0}(t) &= -2t^3 + (x^{*4} - 2x^{*3} - 3x^*)t - 4x^{*4} + 10x^*,\\
\Psi_{x_1 1}(t) &= -2t^3 + (x^{*4} - 2x^{*3} - 3x^*)t - 4x^{*4} + 10x^* + 1,\\
\Psi_{x_1 2}(t) &= 1,
\end{aligned}$$

where we used $F_{x_1(0)}(0) = -4x^{*4} + 10x^* \neq 0$ and $F_{x_1(1)}(0) = -4x^{*4} + 10x^* + 1 \neq 0$ for $0 < x^* \leq 1.3$. Therefore by noting $\Psi_{x_1 0}(0) = -4x^{*4} + 10x^* > 0$ and $\Psi_{x_1 1}(0) = -4x^{*4} + 10x^* + 1 > 0$ for $0 < x^* \leq 1.3$, we have $V\{\Psi_{x_1}(0)\} = 0$. Moreover, for $x^* = 0$, we have $F_{x_1(0)}(0) = 0$, $F_{x_1(0)}(y^*) = f_{(0)}(0, y^*) = -2y^{*3} \neq 0$ for $0 < y^* \leq 1.3$ and

$$\begin{aligned}
\Psi_{x_1 0}(t) &= -2,\\
\Psi_{x_1 1}(t) &= -2t^3 + 1,\\
\Psi_{x_1 2}(t) &= 2,
\end{aligned}$$

which implies $\mathrm{sgn}(\Psi_{x_1 0}(0)\Psi_{x_1 1}(0)) = -1$ and $V\{\Psi_{x_1}(0)\} = V\{\Psi_{x_1}(y^*)\} = 1$. To summarize, we have

$$\Upsilon_{(x^*,y^*)}(1) = \begin{cases} \left.\begin{array}{l} -\arctan\left\{\frac{-4x^{*4}+10x^*+1}{-4x^{*4}+10x^*}\right\} \\ +\begin{cases} \arctan\left\{\frac{f_{(1)}(x^*,y^*)}{f_{(0)}(x^*,y^*)}\right\} + V\{\Psi_{x_1}(y^*)\}\pi & \text{if } f_{(0)}(x^*,y^*) \neq 0, \\ \pi/2 + V\{\Psi_{x_1}(y^*)\}\pi & \text{if } f_{(0)}(x^*,y^*) = 0, \end{cases} \end{array}\right\} & \text{if } 0 < x^* \leq 1.3 \text{ and } 0 < y^* \leq 1.3, \\ \pi/2 + \arctan\left\{\frac{-2y^{*3}+1}{-2y^{*3}}\right\} & \text{if } x^* = 0 \text{ and } 0 < y^* \leq 1.3, \\ 0 & \text{if } y^* = 0. \end{cases}$$

Finally, the unwrapped phase $\theta_f(x^*, y^*)$ is expressed as

$$\theta_f(x^*, y^*) = \pi/2 + \Upsilon_{(x^*,y^*)}(0) + \Upsilon_{(x^*,y^*)}(1)$$

because $f_{(0)}(0,0) = 0$ and $f_{(1)}(0,0) = 1$ imply $\theta_f(0,0) = \pi/2$. Figure 6 depicts the unwrapped phase $\theta_f(x,y)$ for $(x,y) \in [0, 1.3] \times [0, 1.3]$.

4 Stabilizations of the algebraic phase unwrapping along the real axis

4.1 Numerical Instabilities of Algorithm 1

To implement Algorithm 1 (Sturm-$\mathcal{R}$) precisely, we need large number of digits to express the rational coefficients of the polynomials $\Psi_k(t)$ (e.g., see Example 1). This phenomenon is exactly same as the *coefficient growth* well-known in the computation of the *polynomial remainder sequence* through the Euclidean algorithm (Brown and Traub 1971). In computer implementation of $\theta_A(t)$ in Eq. (9) through Algorithm 1, the coefficient growth causes the truncation error in the floating-point expression of the rational coefficients (or memory shortages by increasing number of digits for exact expression of the rational coefficients). In particular, once a serious *information loss* (by the addition or subtraction among numbers of ill-balanced absolute values) or *catastrophic cancellation* (by the subtraction number very close numbers) occurs, the gap between theoretical values and numerical values of $\{\Psi_k(t)\}_{k=0}^{q}$ by digital computer becomes unacceptably large (see Example 7).

Example 7 (Catastrophic cancellation) The Sturm sequence $\{\Psi_k(t)\}_{k=0}^{5}$ obtained in Example 1 is expressed in decimal number expression as

$$\begin{aligned}
\Psi_0(t) &= t^3 - 1.11t^2 + 0.356t - 0.0255,\\
\Psi_1(t) &= t^4 - 2.525t^3 + 2.29995t^2 - 0.906172t + 0.131222,\\
\Psi_2(t) &= -t^3 + 1.11t^2 - 0.356t + 0.0255,\\
\Psi_3(t) &= -0.3733t^2 + 0.376932t - 0.0951395,\\
\Psi_4(t) &= -1.0683812872\ldots \times 10^{-4}t + 5.5025263559\ldots \times 10^{-5},\\
\Psi_5(t) &= 2.8269837842\ldots \times 10^{-5}.
\end{aligned}$$

From Example 1 and the above decimal expression of $\{\Psi_k(t)\}_{k=0}^{5}$, we observe that the absolute value of the coefficients of $\Psi_k(t)$ decreases drastically from $k = 3$ to $k = 4$. It is well-known that such a phenomenon often happens in particular when there exists a close root pair among the roots of $\Psi_0(t)$ and those of $\Psi_1(t)$ (Sasaki and Sasaki 1997, 1989).

In order to deal with the coefficient growth observed widely in the Euclidean algorithm (Brown and Traub 1971), consider the situation where we compute $\{\Psi_k(t)\}_{k=0}^{5}$ with a digital computer based on 64-bit floating point operations. From 1, we expect $|\mathrm{lc}(\Psi_4)| = \dfrac{27788829033}{260102169185000}$ can be approximated in the computer as

$$\begin{aligned}
&\left(\frac{27788829033}{260102169185000}\right)_{f64}\\
&= 1.1100000000011100100100010011010010101000110011001101_2 \times 2^{-14}\\
&= 1.0683812872485099_{10} \times 10^{-4},
\end{aligned}$$

where $(\alpha)_{f64}$ stands for the 64-bit floating point expression of $\alpha \in \mathbb{R}$. Unfortunately, in the process of computation of $\mathrm{lc}(\Psi_4)$ from the coefficients of $\Psi_2(t)$ and of $\Psi_3(t)$, we encounter a severe loss of significant digits (this phenomenon is so-called the *catastrophic cancellation*) as follows.

$$\begin{aligned}
&\left[-(0.356)_{f64} - \left(\frac{(-1)\times(-0.0951395)}{-0.3733}\right)_{f64} - \left(\frac{\left(1.11 - \frac{(-1)\times 0.376932}{-0.3733}\right)_{f64} \times 0.376932}{-0.3733}\right)_{f64}\right]_{f64}\\
&= 1.1100000000011100100100010011010010101101010000000000_2 \times 2^{-14}\\
&= 1.0683812872509801_{10} \times 10^{-4}.
\end{aligned}$$

This fact suggests that it is hard to compute the coefficients of $\{\Psi_k(t)\}_{k=0}^{q}$ by a direct computer implementation of Algorithm 1 (Sturm-$\mathcal{R}$). Indeed, such inaccurate computations of the coefficients lead to failure in counting the sign changes in the Sturm sequence, causing thus numerical instability of the algebraic phase unwrapping.

Once the *information loss* or the *catastrophic cancellation* occurs, this influences inductively in the process of Algorithm 1, which results in the loss of the central property in Lemma 1(c)

$$\Psi_k(t^*) = 0 \text{ at } t^* \in [0,1] \Rightarrow \Psi_{k-1}(t^*)\Psi_{k+1}(t^*) < 0 \quad (k = 1, 2, \ldots, q-1),$$

leading thus to the failure of the computation of (9). This situation restricts the practical applicability of Theorem 1 especially for polynomials of large degree.

4.2 Stabilization of the algebraic phase unwrapping by subresultant sequence

In Section 4.1, we introduced a typical phenomenon causing the considerable gap between theoretical values and numerical values of $\{\Psi_k(t)\}_{k=0}^{q}$ by a digital computer implementation of Algorithm 1 (Sturm-$\mathcal{R}$). In this subsection we present an idea to stabilize the algebraic phase unwrapping along the real axis by replacing Algorithm 1 with new algorithms based on subresultant sequence.

By using Fact 2 repeatedly, we have the following propositions and theorem.

Proposition 4 (Relation between the subresultant sequence and the polynomial remainder sequence) *Let* $\{P_k(t)\}_{k=0}^{q}$ $(q \geq 2)$ *be the polynomial remainder sequence defined inductively as in Remark 3 for* $\deg(P_0) \geq \deg(P_1)$.

(a) *For any* $k \in \{1,2,\ldots,q-1\}$, $\mathrm{Sres}_i(P_0,P_1,t)$ $(i \in [\deg(P_{k+1}),\deg(P_k)-1])$ *can be expressed as*

$$\mathrm{Sres}_i(P_0,P_1,t) = \begin{cases} \lambda_{\deg(P_k)-1} P_{k+1}(t) \\ \qquad \text{for } i = \deg(P_k)-1, \\ 0 \quad \text{for } i \in [\deg(P_{k+1})+1, \deg(P_k)-2] \ \text{(if } \deg(P_{k+1}) < \deg(P_k)-2\text{)}, \\ \lambda_{\deg(P_{k+1})} (\mathrm{lc}(P_{k+1}))^{\deg(P_k)-\deg(P_{k+1})-1} P_{k+1}(t) \\ \qquad \text{for } i = \deg(P_{k+1}), \end{cases} \tag{10}$$

where, for $i = \deg(P_k)-1, \deg(P_{k+1})$,

$$\lambda_i := \prod_{n=0}^{k-2} (-1)^{(\deg(P_n)-\deg(P_{n+1})+1)(\deg(P_{n+1})-i)} (\mathrm{lc}(P_{n+1}))^{\deg(P_n)-\deg(P_{n+2})} \\ \times (-1)^{(\deg(P_{k-1})-\deg(P_k)+1)(\deg(P_k)-i)} (\mathrm{lc}(P_k))^{\deg(P_{k-1})-i} \neq 0. \tag{11}$$

(b) *In particular, if* $\det(M_i(P_0,P_1)) \neq 0$ *for all* $i \in [0,\deg(P_1)-1]$, *we have*

$$\left.\begin{aligned} &\deg(P_{k+1}) = \deg(P_k)-1 = \deg(P_1)-k \\ &\lambda_{\deg(P_k)-1} = \lambda_{\deg(P_{k+1})} = \left((-1)^k \mathrm{lc}(P_1)\right)^{\deg(P_0)-\deg(P_1)+1} \prod_{n=2}^{k} (\mathrm{lc}(P_n))^2 \end{aligned}\right\} \tag{12}$$

for all $k \in [1,q-1]$.

Proof See Appendix 6.

Proposition 5 (Recursive computation of leading coefficients of the polynomial remainder sequence) *Let* $\{P_k(t)\}_{k=0}^{q}$ $(q \geq 2)$ *be the polynomial remainder sequence defined inductively as in Remark 3 for* $\deg(P_0) \geq \deg(P_1)$.

(a) *Suppose that for some* $l \in [1,q-1]$ *the values of* $\deg(P_i)$ *and* $\mathrm{lc}(P_i)$ $(i = 0,1,\ldots,l)$ *are known. Then the values of* $\deg(P_{l+1})$, $\mathrm{lc}(P_{l+1})$ *and* $\mathrm{sgn}(P_{l+1})$ *are obtained as follows.*

$$\deg(P_{l+1}) = \deg(P_l) - \min\{s \in \mathbb{N}^* \mid \det(M_{\deg(P_l)-s}(P_0,P_1)) \neq 0\},$$

$$\mathrm{lc}(P_{l+1}) = \begin{cases} \sqrt[\deg(P_l)-\deg(P_{l+1})]{\dfrac{\det(M_{\deg(P_{l+1})}(P_0,P_1))}{\lambda_{\deg(P_{l+1})}}} \\ \qquad \text{if } (\deg(P_l)-\deg(P_{l+1})) \text{ is odd,} \\ \sqrt[\deg(P_l)-\deg(P_{l+1})-1]{\dfrac{\lambda_{\deg(P_l)-1}\mathrm{Sres}_{\deg(P_{l+1})}(P_0,P_1,\tau)}{\lambda_{\deg(P_{l+1})}\mathrm{Sres}_{\deg(P_l)-1}(P_0,P_1,\tau)}} \\ \qquad \text{(with use of any } \tau\in\mathbb{R} \text{ s.t. } P_{l+1}(\tau)\neq 0) \\ \qquad \text{if } (\deg(P_l)-\deg(P_{l+1})) \text{ is even,} \end{cases}$$

and

$$\mathrm{sgn}(\mathrm{lc}(P_{l+1})) = \begin{cases} \mathrm{sgn}\left(\lambda_{\deg(P_{l+1})}\det(M_{\deg(P_{l+1})}(P_0,P_1))\right) \\ \qquad \text{if } (\deg(P_l)-\deg(P_{l+1})) \text{ is odd,} \\ \mathrm{sgn}\Big(\lambda_{\deg(P_l)-1}\lambda_{\deg(P_{l+1})} \\ \qquad \times\mathrm{Sres}_{\deg(P_l)-1}(P_0,P_1,\tau)\mathrm{Sres}_{\deg(P_{l+1})}(P_0,P_1,\tau)\Big) \\ \qquad \text{(with use of any } \tau\in\mathbb{R} \text{ s.t. } P_{l+1}(\tau)\neq 0) \\ \qquad \text{if } (\deg(P_l)-\deg(P_{l+1})) \text{ is even.} \end{cases}$$

(b) *In particular, if* $\det(M_i(P_0,P_1))\neq 0$ *for all* $i\in[0,\deg(P_1)-1]$*, we have*

$$\left.\begin{aligned} \mathrm{lc}(P_{k+1}) &= \frac{\det(M_{\deg(P_1)-k}(P_0,P_1))}{\left((-1)^k\mathrm{lc}(P_1)\right)^{\deg(P_0)-\deg(P_1)+1}\prod_{i=2}^{k}(\mathrm{lc}(P_i))^2} \\ \mathrm{sgn}(\mathrm{lc}(P_{k+1})) &= \mathrm{sgn}\Big(\left((-1)^k\mathrm{lc}(P_1)\right)^{\deg(P_0)-\deg(P_1)+1}\det(M_{\deg(P_1)-k}(P_0,P_1))\Big) \end{aligned}\right\}$$

for all $k\in[1,q-1]$.

Proof See Appendix 7.

Theorem 3 *(Relation between the sign of the Sturm sequence and the sign of the subresultant sequence) Let* $\{\Psi_k(t)\}_{k=0}^{q}$ *be the Sturm sequence obtained by applying Algorithm 1 to* $A_{(0)}(t)$ *and* $A_{(1)}(t)$*, and let* $\{P_k(t)\}_{k=0}^{q}$ *be the polynomial remainder sequence defined inductively as in Remark 3 for* $P_0(t):=\Psi_0(t)$ *and* $P_1(t):=\Psi_1(t)$*.*

(a) *If* $\deg(\Psi_0)\geq\deg(\Psi_1)$ *and* $q\geq 2$*, we have*

$$\begin{aligned} \mathrm{sgn}(\Psi_k(t^*)) &= (-1)^{\frac{(k-1)k}{2}}\kappa^{\langle 0\rangle}_{\deg(\Psi_k)}\left(\mathrm{sgn}(\mathrm{lc}(P_k))\right)^{\deg(\Psi_{k-1})-\deg(\Psi_k)-1} \\ &\quad\times\mathrm{sgn}\left(\mathrm{Sres}_{\deg(\Psi_k)}(\Psi_0,\Psi_1,t^*)\right) \qquad (k=2,3,\ldots,q), \end{aligned} \tag{13}$$

where

$$\kappa^{\langle 0\rangle}_{\deg(\Psi_k)} := \prod_{n=0}^{k-2}\Big[(-1)^{(\deg(\Psi_n)-\deg(\Psi_{n+1})+1)(\deg(\Psi_{n+1})-\deg(\Psi_k))} \times\left(\mathrm{sgn}(\mathrm{lc}(P_{n+1}))\right)^{\deg(\Psi_n)-\deg(\Psi_{n+2})}\Big].$$

In particular, if $\det(M_i(\Psi_0,\Psi_1)) \neq 0$ *for all* $i \in [0, \deg(\Psi_1)-1]$, *we have* $q = \deg(\Psi_1)+1$ *and*

$$\begin{aligned}\operatorname{sgn}(\Psi_k(t^*)) &= (-1)^{\frac{(k-1)k}{2}+(k-1)(\deg(\Psi_0)-\deg(\Psi_1)+1)} \\ &\quad\times \left(\operatorname{sgn}(\operatorname{lc}(\Psi_1))\right)^{\deg(\Psi_0)-\deg(\Psi_1)+1} \operatorname{sgn}\left(\operatorname{Sres}_{\deg(\Psi_1)-k+1}(\Psi_0,\Psi_1,t^*)\right) \\ &\qquad (k = 2,3,\ldots,\deg(\Psi_1)+1). \quad (14)\end{aligned}$$

(b) *If* $\deg(\Psi_0) < \deg(\Psi_1)$ *and* $q \geq 3$, *we have* $\operatorname{sgn}(\Psi_2(t^*)) = -\operatorname{sgn}(\Psi_0(t^*))$ *and*

$$\begin{aligned}\operatorname{sgn}(\Psi_k(t^*)) &= (-1)^{\frac{(k-1)k}{2}} \kappa^{\langle 1\rangle}_{\deg(\Psi_k)} \left(\operatorname{sgn}(\operatorname{lc}(P_k))\right)^{\deg(\Psi_{k-1})-\deg(\Psi_k)-1} \\ &\quad\times \operatorname{sgn}\left(\operatorname{Sres}_{\deg(\Psi_k)}(\Psi_1,\Psi_0,t^*)\right) \qquad (k = 3,4,\ldots,q), \qquad (15)\end{aligned}$$

where

$$\begin{aligned}\kappa^{\langle 1\rangle}_{\deg(\Psi_k)} := \prod_{n=1}^{k-2} \Big[&(-1)^{(\deg(\Psi_n)-\deg(\Psi_{n+1})+1)(\deg(\Psi_{n+1})-\deg(\Psi_k))} \\ &\times \left(\operatorname{sgn}(\operatorname{lc}(P_{n+1}))\right)^{\deg(\Psi_n)-\deg(\Psi_{n+2})} \Big].\end{aligned}$$

In particular, if $\det(M_i(\Psi_1,\Psi_0)) \neq 0$ *for all* $i \in [0, \deg(\Psi_0)-1]$, *we have* $q = \deg(\Psi_0)+2$ *and*

$$\begin{aligned}\operatorname{sgn}(\Psi_k(t^*)) &= (-1)^{\frac{(k-1)k}{2}+(k-2)(\deg(\Psi_1)-\deg(\Psi_0)+1)} \\ &\quad\times \left(\operatorname{sgn}(\operatorname{lc}(\Psi_0))\right)^{\deg(\Psi_1)-\deg(\Psi_0)+1} \operatorname{sgn}\left(\operatorname{Sres}_{\deg(\Psi_0)-k+2}(\Psi_1,\Psi_0,t^*)\right) \\ &\qquad (k = 3,4,\ldots,\deg(\Psi_0)+2). \quad (16)\end{aligned}$$

Proof See Appendix 8.

The relations (13), (14), (15) and (16) imply that we can compute each sign of $\Psi_k(t^*)$ by $\{\operatorname{Sres}_i(\Psi_0,\Psi_1,t^*)\}_{i=0}^{\deg(\Psi_1)-1}$ (or $\{\operatorname{Sres}_i(\Psi_1,\Psi_0,t^*)\}_{i=0}^{\deg(\Psi_0)-1}$) without computing the coefficients of $\{\Psi(t)\}_{k=0}^{q}$.

Algorithm 2 below evaluates the signs of $\{\Psi_k(t^*)\}_{k=0}^{q}$ based on (14) and (16). In practice, Algorithm 2 plays an adequate role because the condition:

$$\det(M_i(\Psi_0,\Psi_1)) \neq 0 \text{ for all } i \in [0, \min\{\deg(\Psi_0)-1, \deg(\Psi_1)-1\}] \qquad (17)$$

holds almost always. For completeness, we present Algorithms 3 based on (13), (15) and Proposition 5(a) for universal use to evaluate the sign of $\{\Psi_k(t^*)\}_{k=0}^{q}$. Note that the coefficients of $\Psi_k(t) \in \mathbb{R}[t]$ $(k = 2,3,\ldots,q)$ are not necessary for evaluating the signs of $\{\Psi_k(t^*)\}_{k=0}^{q}$ in Algorithm 2 and 3. In Algorithms 2 and 3, we use for simplicity, the following notations: $\deg_k := \deg(\Psi_k)$ and $\operatorname{slc}_k := \operatorname{sgn}(\operatorname{lc}(P_k))$.The computational complexity for each determinant $\operatorname{Sres}_i(\Psi_0,\Psi_1,t^*)$ is at most $\mathcal{O}\left((\deg(\Psi_0)+\deg(\Psi_1)-2i)^{\log_2 7}\right) \approx \mathcal{O}\left((\deg(\Psi_0)+\deg(\Psi_1)-2i)^{2.81}\right)$ (Aho et al. 1974).

Algorithm 2 Proposed algorithm for computing (9) under Condition (17)

Input: $A_{(0)}(t), A_{(1)}(t) \in \mathbb{R}[t]$ and $a, b \in \mathbb{R}$ (s.t. $A_{(0)}(t) + jA_{(1)}(t) \neq 0$ for all $t \in [a,b]$ and $A_{(0)}(t), A_{(1)} \not\equiv 0$)

$\Psi_0(t) \leftarrow \dfrac{A_{(0)}(t)}{(t-a)^{e_0}}$, $\Psi_1(t) \leftarrow \dfrac{A_{(1)}(t)}{(t-a)^{e_1}}$
  (where e_i denotes the order of $t=a$ as a zero of polynomial $A_{(i)}(t)$ $(i=0,1)$)
$\deg_0 \leftarrow \deg(\Psi_0)$, $\deg_1 \leftarrow \deg(\Psi_1)$, $\mathrm{slc}_0 \leftarrow \mathrm{lc}(\Psi_0)$, $\mathrm{slc}_1 \leftarrow \mathrm{lc}(\Psi_1)$
if $\deg_0 \geq \deg_1$ **then**
  for $k = 2$ to $(\deg_1 + 1)$ **do**
    $\mathrm{sgn}(\Psi_k(t^*)) \leftarrow (-1)^{\frac{(k-1)k}{2}+(k-1)(\deg_0-\deg_1+1)}\mathrm{slc}_1^{\deg_0-\deg_1+1}\mathrm{sgn}(\mathrm{Sres}_{\deg_1-k+1}(\Psi_0,\Psi_1,t^*))$
  end for
else
  $\mathrm{sgn}(\Psi_2(t^*)) \leftarrow -\mathrm{sgn}(\Psi_0(t^*))$
  for $k = 3$ to $(\deg_0 + 2)$ **do**
    $\mathrm{sgn}(\Psi_k(t^*)) \leftarrow (-1)^{\frac{(k-1)k}{2}+(k-2)(\deg_1-\deg_0+1)}\mathrm{slc}_0^{\deg_1-\deg_0+1}\mathrm{sgn}(\mathrm{Sres}_{\deg_0-k+1}(\Psi_1,\Psi_0,t^*))$
  end for
end if

Output: $\{\mathrm{sgn}(\Psi_k(t^*))\}_{k=0}^{\min\{\deg_0+2,\deg_1+1\}}$

5 Numerical Example

In this section, we examine the numerical performance of the algebraic phase unwrapping, based on Theorem 1 using Algorithm 2, in 64-bit floating point arithmetic (which has 53 bits of precision) and multiple precision (MP) arithmetic with 80 and 100 bits of precision. (Note: MP arithmetic enables us to compute with designated precision although it takes in general much longer time than the floating point arithmetic). To make the situation likely to cause numerical instability of the algebraic phase unwrapping over $[0,1]$, based on Theorem 1 using Algorithm 1, we generate randomly a pair of polynomials:

$$\left.\begin{aligned} A_{(0)}(t) &:= (t-0.1)(t-0.21)(t-0.5)(t-0.75)(t-0.8)\widehat{A}_{(0)}(t) \\ A_{(1)}(t) &:= (t-0.15)(t-0.2)(t-0.34)(t-0.35)(t-0.81)\widehat{A}_{(1)}(t) \end{aligned}\right\}, \tag{18}$$

where (i) $\widehat{A}_{(0)}(t)$ is a polynomial of degree 35 whose 5 roots are generated by the uniform distribution over $\{(-5,-1)\cup(1,5)\}$ and 15 complex conjugate pairs of roots are generated by the uniform distribution over $\{(-1,-0.5)\cup(0.5,1)\} \pm j\{(-1,-0.5)\cup(0.5,1)\}$, (ii) $\widehat{A}_{(1)}(t)$ is a polynomial of degree 15 whose 5 roots are generated by the uniform distribution over $\{(-5,-1)\cup(1,5)\}$ and 5 complex conjugate pairs of roots are generated by the uniform distribution over $\{(-1,-0.5)\cup(0.5,1)\} \pm j\{(-1,-0.5)\cup(0.5,1)\}$, and (iii) $\mathrm{mmc}(A_{(0)}) = \mathrm{mmc}(A_{(1)}) = 1$. Note that polynomials $A_{(0)}(t)$ and $A_{(1)}(t)$ in (18) have close root pairs $(0.21, 0.8) \approx (0.2, 0.81)$, which likely causes the catastrophic cancellation explained in Sect. 4.1. In this numerical simulation, since all roots of $A_{(0)}(t)$ and $A_{(1)}(t)$ are known, we can compute the exact unwrapped phase by using Eq. (7). Hence we can verify whether Algorithms 1 and 2 succeed or not in phase unwrapping with Eq. (9). Figure 7 depicts one example where Algorithm 1 fails in phase unwrapping at $t = 0.2$ and $t = 0.81$ while Algorithm 2 succeeds in phase unwrapping over $[0,1]$. Table 1 summarizes the result for 1000 trials, where we observe that (i) in 64-bit floating point arithmetic, the total number of pairs of polynomials $(A_{(0)}, A_{(1)})$ in failure by Algorithm 1 is reduced to less than 1/24 by replacing it with Algorithm 2, and (ii) Algorithms 1 and 2, using MP arithmetic with 80 and 100 bits of precision, reduce further the total number of failures.

Algorithm 3 Proposed algorithm for computing (9) in general cases

Input: $A_{(0)}(t), A_{(1)}(t) \in \mathbb{R}[t]$ and $a, b \in \mathbb{R}$ (s.t. $A_{(0)}(t) + jA_{(1)}(t) \neq 0$ for all $t \in [a,b]$ and $A_{(0)}(t), A_{(1)} \not\equiv 0$)

1: $\Psi_0(t) \leftarrow \dfrac{A_{(0)}(t)}{(t-a)^{e_0}}$, $\Psi_1(t) \leftarrow \dfrac{A_{(1)}(t)}{(t-a)^{e_1}}$
(where e_i denotes the order of $t = a$ as a zero of polynomial $A_{(i)}(t)$ $(i = 0, 1)$)

2: $\mathrm{deg}_0 \leftarrow \mathrm{deg}(\Psi_0)$, $\mathrm{deg}_1 \leftarrow \mathrm{deg}(\Psi_1)$, $\mathrm{slc}_0 \leftarrow \mathrm{sgn}(\mathrm{lc}(\Psi_0))$, $\mathrm{slc}_1 \leftarrow \mathrm{sgn}(\mathrm{lc}(\Psi_1))$

3: **if** $\mathrm{deg}_0 \geq \mathrm{deg}_1$ **then**

4: $\quad i \leftarrow 1$, $k \leftarrow 1$

5: $\quad$ **while** $\det(M_{\mathrm{deg}_k - i}(\Psi_0, \Psi_1)) = 0$ and $(\mathrm{deg}_k - i) \geq 1$ **do**

6: $\quad\quad i \leftarrow i + 1$

7: $\quad$ **end while**

8: $\quad$ **while** $(\mathrm{deg}_k - i) \geq 1$ or $\det(M_0(\Psi_0, \Psi_1)) \neq 0$ **do**

9: $\quad\quad \mathrm{deg}_{k+1} \leftarrow \mathrm{deg}_k - i$

10: $\quad\quad \lambda_{\mathrm{deg}_k - 1} \leftarrow \prod_{n=0}^{k-2} (-1)^{(\mathrm{deg}_n - \mathrm{deg}_{n+1} + 1)(\mathrm{deg}_{n+1} - \mathrm{deg}_k + 1)} (\mathrm{slc}_{n+1})^{\mathrm{deg}_n - \mathrm{deg}_{n+2}}$
$\quad\quad\quad\quad \times (-1)^{\mathrm{deg}_{k-1} - \mathrm{deg}_k + 1} (\mathrm{slc}_k)^{\mathrm{deg}_{k-1} - \mathrm{deg}_k + 1}$

11: $\quad\quad \lambda_{\mathrm{deg}_{k+1}} \leftarrow \prod_{n=0}^{k-1} (-1)^{(\mathrm{deg}_n - \mathrm{deg}_{n+1} + 1)(\mathrm{deg}_{n+1} - \mathrm{deg}_{k+1})} (\mathrm{slc}_{n+1})^{\mathrm{deg}_n - \mathrm{deg}_{n+2}}$

12: $\quad\quad$ **if** $(\mathrm{deg}_k - \mathrm{deg}_{k+1})$ is odd **then**

13: $\quad\quad\quad \mathrm{slc}_{k+1} \leftarrow \mathrm{sgn}(\lambda_{\mathrm{deg}_{k+1}} \det(M_{\mathrm{deg}_{k+1}}(\Psi_0, \Psi_1)))$

14: $\quad\quad$ **else**

15: $\quad\quad\quad \mathrm{slc}_{k+1} \leftarrow \mathrm{sgn}(\lambda_{\mathrm{deg}_k - 1} \lambda_{\mathrm{deg}_{k+1}} \mathrm{Sres}_{\mathrm{deg}_k - 1}(\Psi_0, \Psi_1, \tau) \mathrm{Sres}_{\mathrm{deg}_{k+1}}(\Psi_0, \Psi_1, \tau))$

16: $\quad\quad$ **end if**

17: $\quad\quad \mathrm{sgn}(\Psi_{k+1}(t^*)) \leftarrow (-1)^{\frac{k(k+1)}{2}} \lambda_{\mathrm{deg}_{k+1}} \mathrm{slc}_{k+1}^{\mathrm{deg}_k - \mathrm{deg}_{k+1} - 1} \mathrm{sgn}(\mathrm{Sres}_{\mathrm{deg}_{k+1}}(\Psi_0, \Psi_1, t^*))$

18: $\quad\quad i \leftarrow 1$, $k \leftarrow k + 1$

19: $\quad\quad$ **while** $\det(M_{\mathrm{deg}_k - i}(\Psi_0, \Psi_1)) = 0$ and $(\mathrm{deg}_k - i) \geq 1$ **do**

20: $\quad\quad\quad i \leftarrow i + 1$

21: $\quad\quad$ **end while**

22: $\quad$ **end while**

23: **else**

24: $\quad \mathrm{deg}_2 \leftarrow \mathrm{deg}_0$, $\mathrm{slc}_2 \leftarrow \mathrm{slc}_0$, $\mathrm{sgn}(\Psi_2(t^*)) \leftarrow -\mathrm{sgn}(\Psi_0(t^*))$

25: $\quad i \leftarrow 1$, $k \leftarrow 2$

26: $\quad$ **while** $\det(M_{\mathrm{deg}_k - i}(\Psi_1, \Psi_0)) = 0$ and $(\mathrm{deg}_k - i) \geq 1$ **do**

27: $\quad\quad i \leftarrow i + 1$

28: $\quad$ **end while**

29: $\quad$ **while** $(\mathrm{deg}_k - i) \geq 1$ or $\det(M_0(\Psi_1, \Psi_0)) \neq 0$ **do**

30: $\quad\quad \mathrm{deg}_{k+1} \leftarrow \mathrm{deg}_k - i$

31: $\quad\quad \lambda_{\mathrm{deg}_k - 1} \leftarrow \prod_{n=1}^{k-2} (-1)^{(\mathrm{deg}_n - \mathrm{deg}_{n+1} + 1)(\mathrm{deg}_{n+1} - \mathrm{deg}_k + 1)} (\mathrm{slc}_{n+1})^{\mathrm{deg}_n - \mathrm{deg}_{n+2}}$
$\quad\quad\quad\quad \times (-1)^{\mathrm{deg}_{k-1} - \mathrm{deg}_k + 1} (\mathrm{slc}_k)^{\mathrm{deg}_{k-1} - \mathrm{deg}_k + 1}$

32: $\quad\quad \lambda_{\mathrm{deg}_{k+1}} \leftarrow \prod_{n=1}^{k-1} (-1)^{(\mathrm{deg}_n - \mathrm{deg}_{n+1} + 1)(\mathrm{deg}_{n+1} - \mathrm{deg}_{k+1})} (\mathrm{slc}_{n+1})^{\mathrm{deg}_n - \mathrm{deg}_{n+2}}$

33: $\quad\quad$ **if** $(\mathrm{deg}_k - \mathrm{deg}_{k+1})$ is odd **then**

34: $\quad\quad\quad \mathrm{slc}_{k+1} \leftarrow \mathrm{sgn}(\lambda_{\mathrm{deg}_{k+1}} \det(M_{\mathrm{deg}_{k+1}}(\Psi_1, \Psi_0)))$

35: $\quad\quad$ **else**

36: $\quad\quad\quad \mathrm{slc}_{k+1} \leftarrow \mathrm{sgn}(\lambda_{\mathrm{deg}_k - 1} \lambda_{\mathrm{deg}_{k+1}} \mathrm{Sres}_{\mathrm{deg}_k - 1}(\Psi_1, \Psi_0, \tau) \mathrm{Sres}_{\mathrm{deg}_{k+1}}(\Psi_1, \Psi_0, \tau))$

37: $\quad\quad$ **end if**

38: $\quad\quad \mathrm{sgn}(\Psi_{k+1}(t^*)) \leftarrow (-1)^{\frac{k(k+1)}{2}} \lambda_{\mathrm{deg}_{k+1}} \mathrm{slc}_{k+1}^{\mathrm{deg}_k - \mathrm{deg}_{k+1} - 1} \mathrm{sgn}(\mathrm{Sres}_{\mathrm{deg}_{k+1}}(\Psi_1, \Psi_0, t^*))$

39: $\quad\quad i \leftarrow 1$, $k \leftarrow k + 1$

40: $\quad\quad$ **while** $\det(M_{\mathrm{deg}_k - i}(\Psi_1, \Psi_0)) = 0$ and $(\mathrm{deg}_k - i) \geq 1$ **do**

41: $\quad\quad\quad i \leftarrow i + 1$

42: $\quad\quad$ **end while**

43: $\quad$ **end while**

44: **end if**

Output: $\{\mathrm{sgn}(\Psi_k(t^*))\}_{k=0}^{q}$

Remark: In lines 15 and 36: $\tau \in \mathbb{R}$ should satisfy $P_{k+1}(\tau) \neq 0$.

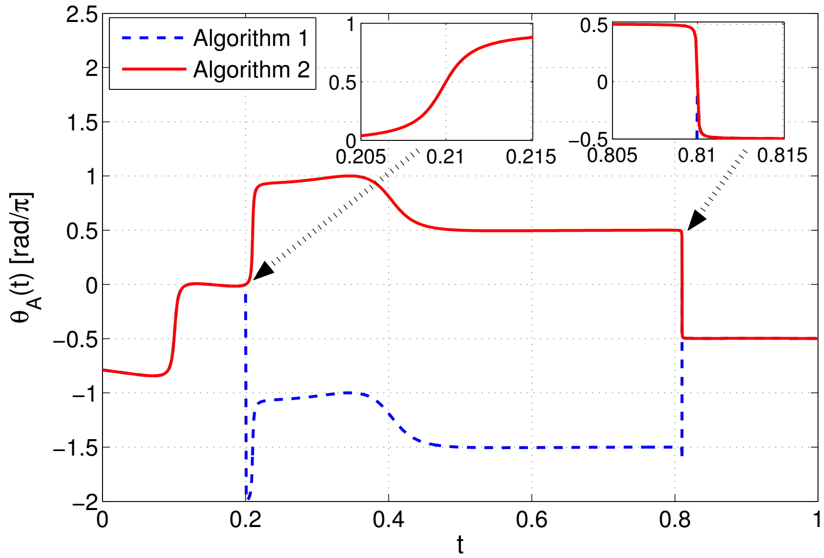


Fig. 7 Estimations of the unwrapped phase with Algorithm 1 and Algorithm 2

Table 1 Performance comparison for pairs of random polynomials

Algorithm	Total number of pairs of polynomials $(A_{(0)}, A_{(1)})$ in failure		
	64-bit floating point	MP with 80 bits of precision	MP with 100 bits of precision
Algorithm 1	249 (among 1000)	9 (among 249)	5 (among 9)
Algorithm 2	10 (among 1000)	7 (among 10)	5 (among 7)

6 Concluding remarks

In this paper, we have extended and stabilized the algebraic phase unwrapping along the real axis. First, by removing a critical assumptions premised in the original algebraic phase unwrapping, we have extended the algebraic phase unwrapping for a pair of piecewise polynomials. Second, we have elucidated the path independence of two-dimensional phase unwrapping completely, and extended the algebraic phase unwrapping for a pair of bivariate polynomials. Third, after clarifying the complete relation between the Sturm sequence, generated by Algorithm 1, and the subresultant sequence, we have shown that the algebraic phase unwrapping along the real axis can be stabilized significantly, by evaluating directly the signs of the Sturm sequence, in the terms of the subresultant sequence.

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Appendices

Appendix 1: On the expression and the integrability of θ_C

Without loss of generality, we can assume $G(t) := \mathrm{GCD}(C(t), \overline{C}(t)) \in \mathbb{R}[t]$ and $B(t) \neq 0$ for all $t \in \mathbb{R}$. Let $\mathcal{Z}_C := \{t \in \mathbb{R} \mid C(t) = 0\}$. Then by $C(t) = G(t)B(t)$, it follows that

$$\frac{C'(t)}{C(t)} = \frac{G'(t)}{G(t)} + \frac{B'(t)}{B(t)} \quad \text{for } t \in \mathbb{R} \setminus \mathcal{Z}_C.$$

Moreover, we have

$$\Im\left\{\frac{C'(t)}{C(t)}\right\} = \Im\left\{\frac{B'(t)}{B(t)}\right\} \quad \text{for all } t \in \mathbb{R} \setminus \mathcal{Z}_C,$$

which, together with $|\mathcal{Z}_C| < \infty$, ensures (3). Furthermore, by the continuity of

$$H(t) := \Im\left\{\frac{B'(t)}{B(t)}\right\} = \frac{B'_{(1)}(t)B_{(0)}(t) - B_{(1)}(t)B'_{(0)}(t)}{\{B_{(0)}(t)\}^2 + \{B_{(1)}(t)\}^2},$$

the integral in (4) is well-defined. □

Appendix 2: Proof of Proposition 1

By

$$(\arctan\{\mathcal{Q}_A(t)\})' = \frac{A'_{(1)}(t)A_{(0)}(t) - A_{(1)}(t)A'_{(0)}(t)}{\{A_{(0)}(t)\}^2 + \{A_{(1)}(t)\}^2}$$

for all $t \in (a,b) \setminus \mathcal{Z}_{A_{(0)}}$ and $|\mathcal{Z}_{A_{(0)}}| < \infty$, we can express the $\theta_A(t^*)$ in (5) in terms of $\arctan\{\mathcal{Q}_A(t)\}$ as follows.

(I) If $\mathcal{Z}_{A_{(0)}} = \varnothing$ or $t^* \leq \mu_1$, we have

$$\begin{aligned}\theta_A(t^*) &= \theta_A(a) + \int_a^{t^*} (\arctan\{\mathcal{Q}_A(t)\})'\,dt \\ &= \theta_A(a) - \lim_{t\to a+0} \arctan\{\mathcal{Q}_A(t)\} + \lim_{t\to t^*-0} \arctan\{\mathcal{Q}_A(t)\}.\end{aligned}$$

and $\Lambda(t^*) = \displaystyle\sum_{\mu_i \in (a,t^*)} \mathcal{X}(\mu_i) = 0$ in (7).

(II) If $\mathcal{Z}_{A_{(0)}} \neq \varnothing$ and $t^* > \mu_1$, by letting $\mu_k := \max(\{\mu_1, \mu_2, \ldots, \mu_z\} \cap [a, t^*))$, we have

$$\begin{aligned}\theta_A(t^*) &= \theta_A(a) + \int_a^{t^*} (\arctan\{\mathcal{Q}_A(t)\})'\,dt \\ &= \theta_A(a) + \int_a^{\mu_1} (\arctan\{\mathcal{Q}_A(t)\})'\,dt \\ &\quad + \sum_{i=1}^{k-1} \int_{\mu_i}^{\mu_{i+1}} (\arctan\{\mathcal{Q}_A(t)\})'\,dt + \int_{\mu_k}^{t^*} (\arctan\{\mathcal{Q}_A(t)\})'\,dt \\ &= \theta_A(a) - \lim_{t\to a+0} \arctan\{\mathcal{Q}_A(t)\} + \lim_{t\to t^*-0} \arctan\{\mathcal{Q}_A(t)\} \\ &\quad + \sum_{i=1}^{k} \lim_{\substack{\tau_1\to\mu_i-0 \\ \tau_2\to\mu_i+0}} (\arctan\{\mathcal{Q}_A(\tau_1)\} - \arctan\{\mathcal{Q}_A(\tau_2)\}).\end{aligned} \tag{19}$$

Furthermore, for μ_i $(i = 1, 2, \ldots, k)$ and sufficiently small $\varepsilon > 0$, we have the following relations.

(i) If $A_{(0)}(t)A_{(1)}(t) > 0$ for $t \in (\mu_i - \varepsilon, \mu_i)$ and $A_{(0)}(t)A_{(1)}(t) < 0$ for $t \in (\mu_i, \mu_i + \varepsilon)$, then

$$\begin{cases} \lim_{t\to\mu_i-0} \arctan\{\mathcal{Q}_A(t)\} = \pi/2, \\ \lim_{t\to\mu_i+0} \arctan\{\mathcal{Q}_A(t)\} = -\pi/2, \end{cases} \quad \text{and} \quad \mathcal{X}(\mu_i) = 1.$$

(ii) If $A_{(0)}(t)A_{(1)}(t) < 0$ for $t \in (\mu_i - \varepsilon, \mu_i)$ and $A_{(0)}(t)A_{(1)}(t) > 0$ for $t \in (\mu_i, \mu_i + \varepsilon)$, then

$$\begin{cases} \lim_{t\to\mu_i-0} \arctan\{\mathcal{Q}_A(t)\} = -\pi/2, \\ \lim_{t\to\mu_i+0} \arctan\{\mathcal{Q}_A(t)\} = \pi/2, \end{cases} \quad \text{and} \quad \mathcal{X}(\mu_i) = -1.$$

(iii) Otherwise,

$$\lim_{t\to\mu_i-0} \arctan\{\mathcal{Q}_A(t)\} = \lim_{t\to\mu_i+0} \arctan\{\mathcal{Q}_A(t)\} = \pm\pi/2, \quad \text{and} \quad \mathcal{X}(\mu_i) = 0.$$

From (i), (ii) and (iii), we have

$$\lim_{\substack{\tau_1\to\mu_i-0 \\ \tau_2\to\mu_i+0}} (\arctan\{\mathcal{Q}_A(\tau_1)\} - \arctan\{\mathcal{Q}_A(\tau_2)\}) = \mathcal{X}(\mu_i)\pi. \tag{20}$$

Finally, (19) and (20) yield (7). □

Appendix 3: Proof of Lemma 1

For the readers' convenience, we present proofs of all statements.

(A) Proof of (a): Assume $\Psi_q(t^*) = 0$ at some $t^* \in [a,b]$. Since $\Psi_q(t)$ is $\mathrm{GCD}(\Psi_0, \Psi_1)$, we have $\Psi_0(t^*) = \Psi_1(t^*) = 0$. Moreover, $\Psi_0(t) := \dfrac{A_{(0)}(t)}{(t-a)^{e_0}}$ and $\Psi_1(t) := \dfrac{A_{(1)}(t)}{(t-a)^{e_1}}$ imply $\Psi_0(a) \neq 0$ and $\Psi_1(a) \neq 0$, and hence, $A_{(0)}(t^*) = \Psi_0(t^*) = \Psi_1(t^*) = A_{(1)}(t^*) = 0$ at some $t^* \in (a,b]$. This contradicts $A(t) = A_{(0)}(t) + jA_{(1)}(t) \neq 0$ for all $t \in [a,b]$. As a result, $\Psi_q(t) \neq 0$ for all $t \in [a,b]$.

(B) Proof of (b): Assume $\Psi_k(t^*) = \Psi_{k+1}(t^*) = 0$ at some $t^* \in [a,b]$. Then $\mathrm{GCD}(\Psi_0, \Psi_1) \equiv \mathrm{GCD}(\Psi_k, \Psi_{k+1})$ implies $\Psi_0(t^*) = \Psi_1(t^*) = 0$, which contradicts $A(t) = A_{(0)}(t) + jA_{(1)}(t) \neq 0$ for all $t \in [a,b]$.

(C) Proof of (c): Suppose $\Psi_k(t^*) = 0$. Then from (b), i.e., $\Psi_{k-1}(t^*), \Psi_{k+1}(t^*) \neq 0$, and $\Psi_{k+1}(t^*) = -\Psi_{k-1}(t^*) + H_k(t^*)\Psi_k(t^*) = -\Psi_{k-1}(t^*)$, we have $\Psi_{k+1}(t^*)\Psi_{k-1}(t^*) < 0$.

(D) Proof of (d): Since $(t-a)^{e_i} > 0$ $(i = 0,1)$ for $a < t \leq b$, the proof is obvious.

(E) Proof of (e): From $\Psi_i(a) \neq 0$ and the continuity of $\Psi_i(t)$ $(i = 0,1)$, we have

$$\lim_{t\to a+0} \mathrm{sgn}(A_{(i)}(t)) = \lim_{t\to a+0} \mathrm{sgn}\left(\frac{A_{(i)}(t)}{(t-a)^{e_i}}\right) = \lim_{t\to a+0} \mathrm{sgn}(\Psi_i(t)) = \mathrm{sgn}(\Psi_i(a)) \neq 0.$$

□

Appendix 4: Proof of Theorem 1

We derive computable expressions for

$$\lim_{t\to a+0} \arctan\{\mathcal{Q}_A(t)\} \quad \text{and} \quad \lim_{t\to t^*-0} \arctan\{\mathcal{Q}_A(t)\} + \Lambda(t^*)\pi$$

in (7) as follows.

(A) Computable expression for $\lim_{t\to a+0} \arctan\{\mathcal{Q}_A(t)\}$ in (7):

(I) If $A_{(0)}(a) \neq 0$, we have $\lim_{t\to a+0} \arctan\{\mathcal{Q}_A(t)\} = \arctan\{\mathcal{Q}_A(a)\}$.

(II) If $A_{(0)}(a) = 0$, then

$$\lim_{t\to a+0} \arctan\{\mathcal{Q}_A(t)\} = \begin{cases} \pi/2 & \text{if } \lim_{t\to a+0} \operatorname{sgn}(A_{(0)}(t)A_{(1)}(t)) = 1, \\ -\pi/2 & \text{if } \lim_{t\to a+0} \operatorname{sgn}(A_{(0)}(t)A_{(1)}(t)) = -1. \end{cases} \tag{21}$$

From Lemma 1(e) and (21), $\lim_{t\to a+0} \arctan\{\mathcal{Q}_A(t)\}$ in (7) can be expressed as

$$\lim_{t\to a+0} \arctan\{\mathcal{Q}_A(t)\} = \operatorname{sgn}(\Psi_0(a)\Psi_1(a))\pi/2.$$

(B) Computable expression for $\lim_{t\to t^*-0} \arctan\{\mathcal{Q}_A(t)\} + \Lambda(t^*)\pi$ in (7):

To derive the relation between $V\{\Psi(t)\}$ and $\mathcal{X}(\mu_i)$ $(i = 1, 2, \ldots, z)$, we have to know the behavior of $V\{\Psi(t)\}$. Since the real polynomials $\Psi_k(t)$ $(0 \leq k \leq q)$ are all continuous, any point where $V\{\Psi(t)\}$ changes must be in the neighborhood of a zero of some $\Psi_k(t)$ $(0 \leq k \leq q)$. Let us observe the behavior of $V\{\Psi(t)\}$ in the neighborhood of a zero of $\Psi_k(t)$ for $0 < k < q$. Suppose

$$\Psi_k(\eta) = 0 \quad \text{for } \eta \in [a, b].$$

From Lemma 1(c) and the continuity of $\Psi_{k-1}(t)$ and $\Psi_{k+1}(t)$, there exists a sufficiently small $\varepsilon > 0$ such that

$$\Psi_{k-1}(t)\Psi_{k+1}(t) < 0 \quad \text{for all } t \in (\eta-\varepsilon, \eta+\varepsilon) \cap [a, b]. \tag{22}$$

From (22), all the possibilities of the sign of $(\Psi_{k-1}(t), \Psi_k(t), \Psi_{k+1}(t))$ in $t \in (\eta - \varepsilon, \eta+\varepsilon) \cap [a, b]$ are $(+, \pm, -)$, $(-, \pm, +)$, $(+, 0, -)$ or $(-, 0, +)$. In all cases, the number of sign changes among $(\Psi_{k-1}(t), \Psi_k(t), \Psi_{k+1}(t))$ is 1. Therefore $V\{\Psi(t)\}$ does not change in the neighborhood of a zero of $\Psi_k(t)$ for $0 < k < q$. Moreover from Lemma 1(a), any change of $V\{\Psi(t)\}$ is caused only by the sign changes of $(\Psi_0(t), \Psi_1(t))$ in the neighborhood of a zero of $\Psi_0(t)$.

To wrap up, for any point ξ_i $(i = 0, 1, \ldots, z)$ such that

$$a \leq \xi_0 < \mu_1 < \xi_1 < \mu_2 < \xi_2 < \cdots < \mu_z < \xi_z < b,$$

we have

$$V\{\Psi(t)\} = \begin{cases} V\{\Psi(\xi_0)\} & \text{if } a \leq t < \mu_1, \\ V\{\Psi(\xi_1)\} & \text{if } \mu_1 < t < \mu_2, \\ \vdots & \\ V\{\Psi(\xi_z)\} & \text{if } \mu_z < t < b. \end{cases} \tag{23}$$

(I) If $\mathcal{Z}_{A_{(0)}} = \varnothing$ or $t^* \in (a, \mu_1)$, we have $\lim\limits_{t \to t^*-0} \arctan\{\mathcal{Q}_A(t)\} = \arctan\{\mathcal{Q}_A(t^*)\}$ and $[V\{\Psi(t^*)\} - V\{\Psi(a)\}]\pi = [V\{\Psi(\xi_0)\} - V\{\Psi(\xi_0)\}]\pi = 0 = \sum\limits_{\mu_i \in (a,t^*)} \mathcal{X}(\mu_i)$.

(II) If $\mathcal{Z}_{A_{(0)}} \neq \varnothing$, $\mu_1 < t^* < b$ and $t^* \neq \mu_i$ $(i = 1, 2, \ldots, z)$, Lemma 1(b) and the continuity of $\Psi_1(t)$ ensure the existence of a sufficiently small $\varepsilon > 0$ for μ_i such that

$$\left.\begin{array}{l} [\mu_i - \varepsilon, \mu_i + \varepsilon] \subset (a, b) \\ \Psi_1(t) \neq 0 \ \text{ for all } t \in (\mu_i - \varepsilon, \mu_i + \varepsilon) \\ \Psi_0(t)\Psi_1(t) \neq 0 \ \text{ for all } t \in (\mu_i - \varepsilon, \mu_i) \cup (\mu_i, \mu_i + \varepsilon) \end{array}\right\}. \tag{24}$$

We fix arbitrarily $\xi_{i-1} \in (\mu_i - \varepsilon, \mu_i)$ and $\xi_i' \in (\mu_i, \mu_i + \varepsilon)$.

(i) If $A_{(0)}(t)A_{(1)}(t) > 0 \big(\overset{\text{Lemma 1(d)}}{\Longleftrightarrow} \Psi_0(t)\Psi_1(t) > 0\big)$ for $t \in (\mu_i - \varepsilon, \mu_i)$ and $A_{(0)}(t)A_{(1)}(t) < 0 \big(\overset{\text{Lemma 1(d)}}{\Longleftrightarrow} \Psi_0(t)\Psi_1(t) < 0\big)$ for $t \in (\mu_i, \mu_i + \varepsilon)$, we have $\operatorname{sgn}(\Psi_0(\xi_{i-1})) = \operatorname{sgn}(\Psi_1(\xi_{i-1}))$ and $\operatorname{sgn}(\Psi_0(\xi_i')) = -\operatorname{sgn}(\Psi_1(\xi_i'))$. Then the number of sign changes of $(\Psi_0(\xi_{i-1}), \Psi_1(\xi_{i-1}))$ is 0, and that of $(\Psi_0(\xi_i'), \Psi_1(\xi_i'))$ is 1. Moreover by (23), we have $V\{\Psi(\xi_i')\} = V\{\Psi(\xi_i)\}$. Hence, we have

$$V\{\Psi(\xi_i)\} - V\{\Psi(\xi_{i-1})\} = V\{\Psi(\xi_i')\} - V\{\Psi(\xi_{i-1})\} = 1 \ \text{ and } \ \mathcal{X}(\mu_i) = 1.$$

(ii) If $A_{(0)}(t)A_{(1)}(t) < 0 \big(\overset{\text{Lemma 1(d)}}{\Longleftrightarrow} \Psi_0(t)\Psi_1(t) < 0\big)$ for $t \in (\mu_i - \varepsilon, \mu_i)$ and $A_{(0)}(t)A_{(1)}(t) > 0 \big(\overset{\text{Lemma 1(d)}}{\Longleftrightarrow} \Psi_0(t)\Psi_1(t) > 0\big)$ for $t \in (\mu_i, \mu_i + \varepsilon)$, we have $\operatorname{sgn}(\Psi_0(\xi_{i-1})) = -\operatorname{sgn}(\Psi_1(\xi_{i-1}))$ and $\operatorname{sgn}(\Psi_0(\xi_i')) = \operatorname{sgn}(\Psi_1(\xi_i'))$. Then the number of sign changes of $(\Psi_0(\xi_{i-1}), \Psi_1(\xi_{i-1}))$ is 1, and that of $(\Psi_0(\xi_i'), \Psi_1(\xi_i'))$ is 0. Moreover by (23), we have $V\{\Psi(\xi_i')\} = V\{\Psi(\xi_i)\}$. Hence, we have

$$V\{\Psi(\xi_i)\} - V\{\Psi(\xi_{i-1})\} = V\{\Psi(\xi_i')\} - V\{\Psi(\xi_{i-1})\} = -1 \ \text{ and } \ \mathcal{X}(\mu_i) = -1.$$

(iii) If $A_{(0)}(t)A_{(1)}(t) > 0 \big(\overset{\text{Lemma 1(d)}}{\Longleftrightarrow} \Psi_0(t)\Psi_1(t) > 0\big)$ for $t \in (\mu_i - \varepsilon, \mu_i)$ and $A_{(0)}(t)A_{(1)}(t) > 0 \big(\overset{\text{Lemma 1(d)}}{\Longleftrightarrow} \Psi_0(t)\Psi_1(t) > 0\big)$ for $t \in (\mu_i, \mu_i + \varepsilon)$, we have $\operatorname{sgn}(\Psi_0(\xi_{i-1})) = \operatorname{sgn}(\Psi_1(\xi_{i-1}))$ and $\operatorname{sgn}(\Psi_0(\xi_i')) = \operatorname{sgn}(\Psi_1(\xi_i'))$. Then the number of sign changes of $(\Psi_0(\xi_{i-1}), \Psi_1(\xi_{i-1}))$ is 0, and that of $(\Psi_0(\xi_i'), \Psi_1(\xi_i'))$ is 0. Moreover by (23), we have $V\{\Psi(\xi_i')\} = V\{\Psi(\xi_i)\}$. Hence, we have

$$V\{\Psi(\xi_i)\} - V\{\Psi(\xi_{i-1})\} = V\{\Psi(\xi_i')\} - V\{\Psi(\xi_{i-1})\} = 0 \ \text{ and } \ \mathcal{X}(\mu_i) = 0.$$

(iv) If $A_{(0)}(t)A_{(1)}(t) < 0 \big(\overset{\text{Lemma 1(d)}}{\Longleftrightarrow} \Psi_0(t)\Psi_1(t) < 0\big)$ for $t \in (\mu_i - \varepsilon, \mu_i)$ and $A_{(0)}(t)A_{(1)}(t) < 0 \big(\overset{\text{Lemma 1(d)}}{\Longleftrightarrow} \Psi_0(t)\Psi_1(t) < 0\big)$ for $t \in (\mu_i, \mu_i + \varepsilon)$, we have $\operatorname{sgn}(\Psi_0(\xi_{i-1})) = -\operatorname{sgn}(\Psi_1(\xi_{i-1}))$ and $\operatorname{sgn}(\Psi_0(\xi_i')) = -\operatorname{sgn}(\Psi_1(\xi_i'))$. Then the number of sign changes of $(\Psi_0(\xi_{i-1}), \Psi_1(\xi_{i-1}))$ is 1, and that of $(\Psi_0(\xi_i'), \Psi_1(\xi_i'))$ is 1. Moreover by (23), we have $V\{\Psi(\xi_i')\} = V\{\Psi(\xi_i)\}$. Hence, we have

$$V\{\Psi(\xi_i)\} - V\{\Psi(\xi_{i-1})\} = V\{\Psi(\xi_i')\} - V\{\Psi(\xi_{i-1})\} = 0 \ \text{ and } \ \mathcal{X}(\mu_i) = 0.$$

As a result, in all cases (i), (ii), (iii) and (iv), we have

$$V\{\Psi(\xi_i)\} - V\{\Psi(\xi_{i-1})\} = \mathcal{X}(\mu_i).$$

Finally, by letting $\mu_k := \max(\{\mu_1, \mu_2, \ldots, \mu_z\} \cap [a, t^*))$ in the definition of $\Lambda(t^*)$, we have, from $A_{(0)}(t^*) \neq 0$ and (23),

$$\begin{aligned}
&\lim_{t \to t^* - 0} \arctan\{\mathcal{Q}_A(t)\} + \Lambda(t^*)\pi \\
&\qquad = \arctan\{\mathcal{Q}_A(t^*)\} + \sum_{i=1}^{k} \mathcal{X}(\mu_i)\pi \\
&\qquad = \arctan\{\mathcal{Q}_A(t^*)\} + \sum_{i=1}^{k} [V\{\Psi(\xi_i)\} - V\{\Psi(\xi_{i-1})\}]\pi \\
&\qquad = \arctan\{\mathcal{Q}_A(t^*)\} + [V\{\Psi(\xi_k)\} - V\{\Psi(\xi_0)\}]\pi \\
&\qquad = \arctan\{\mathcal{Q}_A(t^*)\} + [V\{\Psi(t^*)\} - V\{\Psi(a)\}]\pi.
\end{aligned}$$

(III) If $\mathcal{Z}_{A_{(0)}} \neq \varnothing$ and $t^* = \mu_k$ for some $k \in \{1, 2, \ldots, z\}$, Lemma 1(b) and (d) ensure $\operatorname{sgn}(\Psi_1(\mu_k)) = \operatorname{sgn}(A_{(1)}(\mu_k)) \neq 0$, and we have

$$\lim_{t \to t^* - 0} \arctan\{\mathcal{Q}_A(t)\} = \begin{cases} \pi/2 & \text{if } \lim_{t \to \mu_k - 0} \operatorname{sgn}(A_{(0)}(t) A_{(1)}(\mu_k)) = 1, \\ -\pi/2 & \text{if } \lim_{t \to \mu_k - 0} \operatorname{sgn}(A_{(0)}(t) A_{(1)}(\mu_k)) = -1. \end{cases} \tag{25}$$

Fix ξ_{i-1} and ξ_i' $(i = 1, 2, \ldots, k)$ in exactly same way as shown in the beginning of (II). Then we have $V\{\Psi(\xi_i)\} - V\{\Psi(\xi_{i-1})\} = \mathcal{X}(\mu_i)$ $(i = 1, 2, \ldots, k-1)$.

(i) If $\lim_{t \to t^* - 0} \arctan\{\mathcal{Q}_A(t)\} = \pi/2$, Lemma 1(d) and (25) ensure $\operatorname{sgn}(\Psi_0(\xi_{k-1})) = \operatorname{sgn}(\Psi_1(\xi_{k-1})) \neq 0$ and $\operatorname{sgn}(\Psi_0(\mu_k)) = 0$, which imply that both of the numbers of sign changes in $(\Psi_0(\xi_{k-1}), \Psi_1(\xi_{k-1}))$ and in $(\Psi_0(\mu_k), \Psi_1(\mu_k))$ are 0. Hence, we have

$$V\{\Psi(\mu_k)\} = V\{\Psi(\xi_{k-1})\}.$$

As a result, we have

$$\begin{aligned}
\lim_{t \to t^* - 0} \arctan\{\mathcal{Q}_A(t)\} + \Lambda(t^*)\pi &= \pi/2 + \sum_{i=1}^{k-1} \mathcal{X}(\mu_i)\pi \\
&= \pi/2 + \sum_{i=1}^{k-1} [V\{\Psi(\xi_i)\} - V\{\Psi(\xi_{i-1})\}]\pi \\
&= \pi/2 + [V\{\Psi(\xi_{k-1})\} - V\{\Psi(\xi_0)\}]\pi \\
&= \pi/2 + [V\{\Psi(\mu_k)\} - V\{\Psi(a)\}]\pi. \qquad (26)
\end{aligned}$$

(ii) If $\lim_{t \to t^* - 0} \arctan\{\mathcal{Q}_A(t)\} = -\pi/2$, Lemma 1(d) and (25) ensure $\operatorname{sgn}(\Psi_0(\xi_{k-1})) = -\operatorname{sgn}(\Psi_1(\xi_{k-1})) \neq 0$ and $\operatorname{sgn}(\Psi_0(\mu_k)) = 0$, which imply that the number of sign changes in $(\Psi_0(\xi_{k-1}), \Psi_1(\xi_{k-1}))$ is 1 while the number of sign changes in $(\Psi_0(\mu_k), \Psi_1(\mu_k))$ is 0. Hence, we have

$$V\{\Psi(\mu_k)\} = V\{\Psi(\xi_{k-1})\} - 1.$$

As a result, we have

$$\begin{aligned}\lim_{t\to t^*-0}\arctan\{\mathcal{Q}_A(t)\}+\Lambda(t^*)\pi &= -\pi/2+\sum_{i=1}^{k-1}\mathcal{X}(\mu_i)\pi\\ &= -\pi/2+\sum_{i=1}^{k-1}[V\{\Psi(\xi_i)\}-V\{\Psi(\xi_{i-1})\}]\pi\\ &= -\pi/2+[V\{\Psi(\xi_{k-1})\}-V\{\Psi(\xi_0)\}]\pi\\ &= -\pi/2+[V\{\Psi(\mu_k)\}+1-V\{\Psi(a)\}]\pi\\ &= \pi/2+[V\{\Psi(\mu_k)\}-V\{\Psi(a)\}]\pi. \qquad (27)\end{aligned}$$

From (26) and (27), for both cases (i) and (ii), we have

$$\lim_{t\to t^*-0}\arctan\{\mathcal{Q}_A(t)\}+\Lambda(t^*)\pi=\pi/2+[V\{\Psi(t^*)\}-V\{\Psi(a)\}]\pi.$$

(IV) Suppose that $\mathcal{Z}_{A_{(0)}}\neq\varnothing$ and $t^*=b$.

(i) If $A_{(0)}(b)\neq 0$, i.e., $V\{\Psi(\xi_z)\}=V\{\Psi(b)\}$, we can deduce, in almost same way as in proof for (II),

$$\lim_{t\to t^*-0}\arctan\{\mathcal{Q}_A(t)\}+\Lambda(t^*)\pi=\arctan\{\mathcal{Q}_A(t^*)\}+[V\{\Psi(t^*)\}-V\{\Psi(a)\}]\pi.$$

(ii) If $A_{(0)}(b)=0$, by imposing additionally, to the conditions of $\varepsilon>0$ in (24),

$$\left.\begin{array}{l} b-\varepsilon>a\\ \Psi_1(t)\neq 0\ \text{ for all } t\in(b-\varepsilon,b]\\ \Psi_0(t)\Psi_1(t)\neq 0\ \text{ for all } t\in(b-\varepsilon,b)\end{array}\right\}$$

and fixing arbitrarily $\xi_z\in(b-\varepsilon,b)$, we can deduce, in almost same way as in proof for (III),

$$\lim_{t\to t^*-0}\arctan\{\mathcal{Q}_A(t)\}+\Lambda(t^*)\pi=\pi/2+[V\{\Psi(t^*)\}-V\{\Psi(a)\}]\pi.$$

From (A) and (B), we obtain (9) for all $t^*\in(a,b]$. □

Appendix 5: Proof of Theorem 2

(A) Proof of (a): From

$$\Im\left\{\frac{\frac{\partial f_{(0)}}{\partial y}(x,y)+j\frac{\partial f_{(1)}}{\partial y}(x,y)}{f_{(0)}(x,y)+jf_{(1)}(x,y)}\right\}=\frac{\left(\frac{\partial f_{(1)}}{\partial y}(x,y)\right)f_{(0)}(x,y)-f_{(1)}(x,y)\left(\frac{\partial f_{(0)}}{\partial y}(x,y)\right)}{\{f_{(0)}(x,y)\}^2+\{f_{(1)}(x,y)\}^2},$$

the denominator of $\frac{\partial}{\partial x}\left(\Im\left\{\frac{\frac{\partial f_{(0)}}{\partial y}(x,y)+j\frac{\partial f_{(1)}}{\partial y}(x,y)}{f_{(0)}(x,y)+jf_{(1)}(x,y)}\right\}\right)$ is $\left[\{f_{(0)}(x,y)\}^2+\{f_{(1)}(x,y)\}^2\right]^2$, and the numerator is

$$\begin{aligned}&\left[\left(\frac{\partial^2 f_{(1)}}{\partial x\partial y}(x,y)\right)f_{(0)}(x,y)-f_1(x,y)\left(\frac{\partial^2 f_{(0)}}{\partial x\partial y}(x,y)\right)\right]\left[\{f_{(0)}(x,y)\}^2+\{f_{(1)}(x,y)\}^2\right]\\ &-\left[\left(\frac{\partial f_{(1)}}{\partial y}(x,y)\right)\left(\frac{\partial f_{(0)}}{\partial x}(x,y)\right)+\left(\frac{\partial f_{(1)}}{\partial x}(x,y)\right)\left(\frac{\partial f_{(0)}}{\partial y}(x,y)\right)\right]\left[\{f_{(0)}(x,y)\}^2-\{f_{(1)}(x,y)\}^2\right]\\ &-2\left[\left(\frac{\partial f_{(1)}}{\partial x}(x,y)\right)\left(\frac{\partial f_{(1)}}{\partial y}(x,y)\right)-\left(\frac{\partial f_{(0)}}{\partial x}(x,y)\right)\left(\frac{\partial f_{(0)}}{\partial y}(x,y)\right)\right]f_{(0)}(x,y)f_{(1)}(x,y).\end{aligned}$$

Similarly, from

$$\Im\left\{\frac{\frac{\partial f_{(0)}}{\partial x}(x,y)+j\frac{\partial f_{(1)}}{\partial x}(x,y)}{f_{(0)}(x,y)+jf_{(1)}(x,y)}\right\}=\frac{\left(\frac{\partial f_{(1)}}{\partial x}(x,y)\right)f_{(0)}(x,y)-f_{(1)}(x,y)\left(\frac{\partial f_{(0)}}{\partial x}(x,y)\right)}{\{f_{(0)}(x,y)\}^2+\{f_{(1)}(x,y)\}^2},$$

the denominator of $\frac{\partial}{\partial y}\left(\Im\left\{\frac{\frac{\partial f_{(0)}}{\partial x}(x,y)+j\frac{\partial f_{(1)}}{\partial x}(x,y)}{f_{(0)}(x,y)+jf_{(1)}(x,y)}\right\}\right)$ is $\left[\{f_{(0)}(x,y)\}^2+\{f_{(1)}(x,y)\}^2\right]^2$, and the numerator is

$$\begin{aligned}&\left[\left(\frac{\partial^2 f_{(1)}}{\partial y\partial x}(x,y)\right)f_{(0)}(x,y)-f_1(x,y)\left(\frac{\partial^2 f_{(0)}}{\partial y\partial x}(x,y)\right)\right]\left[\{f_{(0)}(x,y)\}^2+\{f_{(1)}(x,y)\}^2\right]\\
&-\left[\left(\frac{\partial f_{(1)}}{\partial x}(x,y)\right)\left(\frac{\partial f_{(0)}}{\partial y}(x,y)\right)+\left(\frac{\partial f_{(1)}}{\partial y}(x,y)\right)\left(\frac{\partial f_{(0)}}{\partial x}(x,y)\right)\right]\left[\{f_{(0)}(x,y)\}^2-\{f_{(1)}(x,y)\}^2\right]\\
&-2\left[\left(\frac{\partial f_{(1)}}{\partial y}(x,y)\right)\left(\frac{\partial f_{(1)}}{\partial x}(x,y)\right)-\left(\frac{\partial f_{(0)}}{\partial y}(x,y)\right)\left(\frac{\partial f_{(0)}}{\partial x}(x,y)\right)\right]f_{(0)}(x,y)f_{(1)}(x,y).\end{aligned}$$

Then, since $f_{(i)}\in C^2(D)$ $(i=0,1)$ ensure $\frac{\partial^2 f_{(i)}}{\partial x\partial y}(x,y)=\frac{\partial^2 f_{(i)}}{\partial y\partial x}(x,y)$ for all $(x,y)\in D$, we have

$$\frac{\partial}{\partial x}\left[\Im\left\{\frac{\frac{\partial f_{(0)}}{\partial y}(x,y)+j\frac{\partial f_{(1)}}{\partial y}(x,y)}{f_{(0)}(x,y)+jf_{(1)}(x,y)}\right\}\right]=\frac{\partial}{\partial y}\left[\Im\left\{\frac{\frac{\partial f_{(0)}}{\partial x}(x,y)+j\frac{\partial f_{(1)}}{\partial x}(x,y)}{f_{(0)}(x,y)+jf_{(1)}(x,y)}\right\}\right]$$

for all $(x,y)\in D$.

(B) Proof of (b): Define

$$P(x,y):=\Im\left\{\frac{\frac{\partial f_{(0)}}{\partial x}(x,y)+j\frac{\partial f_{(1)}}{\partial x}(x,y)}{f_{(0)}(x,y)+jf_{(1)}(x,y)}\right\}\text{ and }Q(x,y):=\Im\left\{\frac{\frac{\partial f_{(0)}}{\partial y}(x,y)+j\frac{\partial f_{(1)}}{\partial y}(x,y)}{f_{(0)}(x,y)+jf_{(1)}(x,y)}\right\}.$$

Then, since $f_{(i)}$ $(i=0,1)$ are $C^2(D)$ functions, P and Q are $C^1(D)$ functions. Moreover, from (a), P and Q satisfy $\frac{\partial P}{\partial y}(x,y)=\frac{\partial Q}{\partial x}(x,y)$ for all $(x,y)\in D$. Hence, from Poincaré's lemma (Fact 1(b)), there exists a function $\theta_f\in C^2(D)$ satisfying

$$\frac{\partial\theta_f}{\partial x}(x,y)=P(x,y)\quad\text{and}\quad\frac{\partial\theta_f}{\partial y}(x,y)=Q(x,y)\quad\text{for all }(x,y)\in D,\tag{28}$$

and the function θ_f is the scalar potential of the vector field $(P(x,y),Q(x,y))$ over D. Eq. (28) implies that the function θ_f is determined as

$$\theta_f(x,y)=\int[P(x,y)dx+Q(x,y)dy]$$

uniquely if we impose additionally the condition $\theta_f(x_0,y_0)=\theta_0$.

(C) Proof of (c): Define $P(x,y)$ and $Q(x,y)$ as in (B). From (a), i.e., $\frac{\partial P}{\partial y}(x,y)=\frac{\partial Q}{\partial x}(x,y)$ for all $(x,y)\in D$, and Green's theorem (Fact 1(a)), we have

$$\oint_{\partial\Omega}[P(x,y)dx+Q(x,y)dy]=\iint_{\Omega}\left(\frac{\partial Q}{\partial x}(x,y)-\frac{\partial P}{\partial y}(x,y)\right)dxdy=0.$$

In particular, if γ^{I} and γ^{II} are piecewise C^1 paths in D with the same initial and final points, by letting $\partial\Omega := \gamma^{\mathrm{I}} - \gamma^{\mathrm{II}}$, we have

$$\oint_{\gamma^{\mathrm{I}}-\gamma^{\mathrm{II}}} [P(x,y)dx + Q(x,y)dy] = 0,$$

which implies

$$\int_{\gamma^{\mathrm{I}}} [P(x,y)dx + Q(x,y)dy] = \int_{\gamma^{\mathrm{II}}} [P(x,y)dx + Q(x,y)dy].$$

(D) Proof of (d): By using the parameterizations $\gamma^{\mathrm{I}}(t) := (x_{\mathrm{I}}(t), y_{\mathrm{I}}(t))$ and $\gamma^{\mathrm{II}}(\tau) := (x_{\mathrm{II}}(\tau), y_{\mathrm{II}}(\tau))$, we deduce, from (c),

$$\begin{aligned}
&\int_a^b \Im\left\{\frac{\left(f_{(0)}(\gamma^{\mathrm{I}}(t))\right)' + j\left(f_{(1)}(\gamma^{\mathrm{I}}(t))\right)'}{f_{(0)}(\gamma^{\mathrm{I}}(t)) + jf_{(1)}(\gamma^{\mathrm{I}}(t))}\right\}dt \\
&= \int_a^b \frac{\frac{d}{dt}\left(f_{(1)}(x_{\mathrm{I}}(t),y_{\mathrm{I}}(t))\right) f_{(0)}(x_{\mathrm{I}}(t),y_{\mathrm{I}}(t)) - f_{(1)}(x_{\mathrm{I}}(t),y_{\mathrm{I}}(t))\frac{d}{dt}\left(f_{(0)}(x_{\mathrm{I}}(t),y_{\mathrm{I}}(t)\right)}{\{f_{(0)}(x_{\mathrm{I}}(t),y_{\mathrm{I}}(t))\}^2 + \{f_{(1)}(x_{\mathrm{I}}(t),y_{\mathrm{I}}(t))\}^2}dt \\
&= \int_{\gamma^{\mathrm{I}}(a)}^{\gamma^{\mathrm{I}}(b)} \left[\frac{\left(\frac{\partial f_{(1)}}{\partial x}(x,y)\right) f_{(0)}(x,y) - f_{(1)}(x,y)\left(\frac{\partial f_{(0)}}{\partial x}(0)(x,y)\right)}{\{f_{(0)}(x,y)\}^2 + \{f_{(1)}(x,y)\}^2}dx \right.\\
&\qquad\qquad \left. + \frac{\left(\frac{\partial f_{(1)}}{\partial y}(x,y)\right) f_{(0)}(x,y) - f_{(1)}(x,y)\left(\frac{\partial f_{(0)}}{\partial y}(x,y)\right)}{\{f_{(0)}(x,y)\}^2 + \{f_{(1)}(x,y)\}^2}dy\right] \\
&= \int_{\gamma^{\mathrm{II}}(c)}^{\gamma^{\mathrm{II}}(d)} \left[\frac{\left(\frac{\partial f_{(1)}}{\partial x}(x,y)\right) f_{(0)}(x,y) - f_{(1)}(x,y)\left(\frac{\partial f_{(0)}}{\partial x}(0)(x,y)\right)}{\{f_{(0)}(x,y)\}^2 + \{f_{(1)}(x,y)\}^2}dx \right.\\
&\qquad\qquad \left. + \frac{\left(\frac{\partial f_{(1)}}{\partial y}(x,y)\right) f_{(0)}(x,y) - f_{(1)}(x,y)\left(\frac{\partial f_{(0)}}{\partial y}(x,y)\right)}{\{f_{(0)}(x,y)\}^2 + \{f_{(1)}(x,y)\}^2}dy\right] \\
&= \int_c^d \frac{\frac{d}{d\tau}\left(f_{(1)}(x_{\mathrm{II}}(\tau),y_{\mathrm{I}}(\tau))\right) f_{(0)}(x_{\mathrm{II}}(\tau),y_{\mathrm{II}}(\tau)) - f_{(1)}(x_{\mathrm{II}}(\tau),y_{\mathrm{I}}(\tau))\frac{d}{d\tau}\left(f_{(0)}(x_{\mathrm{II}}(\tau),y_{\mathrm{II}}(\tau)\right)}{\{f_{(0)}(x_{\mathrm{II}}(\tau),y_{\mathrm{II}}(\tau))\}^2 + \{f_{(1)}(x_{\mathrm{II}}(\tau),y_{\mathrm{II}}(\tau))\}^2}d\tau \\
&= \int_c^d \Im\left\{\frac{\left(f_{(0)}(\gamma^{\mathrm{II}}(\tau))\right)' + j\left(f_{(1)}(\gamma^{\mathrm{II}}(\tau))\right)'}{f_{(0)}(\gamma^{\mathrm{II}}(\tau)) + jf_{(1)}(\gamma^{\mathrm{II}}(\tau))}\right\}d\tau.
\end{aligned}$$

□

Appendix 6: Proof of Proposition 4

(A) Proof of (a):

(I) If $k = 1$, $\mathrm{Sres}_i(P_0, P_1, t)$ ($i \in [\deg(P_2), \deg(P_1) - 1]$) can be expressed as a constant multiple of $P_2(t)$ as one of the first three expressions in Fact 2. Clearly, these are special cases of (10).

(II) If $2 \le k \le q-1$, i.e., $\deg(P_{k+1}) \le i \le \deg(P_k) - 1 \le \deg(P_{k-1}) - 2 \le \deg(P_{k-2}) - 3 \le \cdots \le \deg(P_2) - (k-1)$, by using the forth expression in Fact 2 repeatedly, we deduce

$$
\begin{aligned}
&\mathrm{Sres}_i(P_0,P_1,t)\\
&= (-1)^{(\deg(P_0)-\deg(P_1)+1)(\deg(P_1)-i)}(\mathrm{lc}(P_1))^{\deg(P_0)-\deg(P_2)}\\
&\quad \times \mathrm{Sres}_i(P_1,P_2,t)\\
&= (-1)^{(\deg(P_0)-\deg(P_1)+1)(\deg(P_1)-i)}(\mathrm{lc}(P_1))^{\deg(P_0)-\deg(P_2)}\\
&\quad \times (-1)^{(\deg(P_1)-\deg(P_2)+1)(\deg(P_2)-i)}(\mathrm{lc}(P_2))^{\deg(P_1)-\deg(P_3)}\\
&\quad \times \mathrm{Sres}_i(P_2,P_3,t)\\
&= \cdots\\
&= \prod_{n=0}^{k-2}(-1)^{(\deg(P_n)-\deg(P_{n+1})+1)(\deg(P_{n+1})-i)}(\mathrm{lc}(P_{n+1}))^{\deg(P_n)-\deg(P_{n+2})}\\
&\quad \times \mathrm{Sres}_i(P_{k-1},P_k,t)\\
&= \prod_{n=0}^{k-2}(-1)^{(\deg(P_n)-\deg(P_{n+1})+1)(\deg(P_{n+1})-i)}(\mathrm{lc}(P_{n+1}))^{\deg(P_n)-\deg(P_{n+2})}\\
&\quad \times \begin{cases} (-1)^{\deg(P_{k-1})-\deg(P_k)+1}(\mathrm{lc}(P_k))^{\deg(P_{k-1})-\deg(P_k)+1}P_{k+1}(t) \\ \qquad \text{for } i=\deg(P_k)-1, \\ 0 \qquad \text{for } i\in[\deg(P_{k+1})+1,\deg(P_k)-2] \quad (\text{if } \deg(P_{k+1})<\deg(P_k)-2), \\ (-1)^{(\deg(P_{k-1})-\deg(P_k)+1)(\deg(P_k)-\deg(P_{k+1}))}(\mathrm{lc}(P_k))^{\deg(P_{k-1})-\deg(P_{k+1})} \\ \quad \times (\mathrm{lc}(P_{k+1}))^{\deg(P_k)-\deg(P_{k+1})-1}P_{k+1}(t) \\ \qquad \text{for } i=\deg(P_{k+1}), \end{cases}\\
&= \begin{cases} \lambda_{\deg(P_k)-1}P_{k+1}(t) \\ \qquad \text{for } i=\deg(P_k)-1, \\ 0 \qquad \text{for } i\in[\deg(P_{k+1})+1,\deg(P_k)-2] \quad (\text{if } \deg(P_{k+1})<\deg(P_k)-2), \\ \lambda_{\deg(P_{k+1})}(\mathrm{lc}(P_{k+1}))^{\deg(P_k)-\deg(P_{k+1})-1}P_{k+1}(t) \\ \qquad \text{for } i=\deg(P_{k+1}). \end{cases}
\end{aligned}
$$

(B) Proof of (b): If $\det(M_i(P_0,P_1)) \neq 0$ for all $i \in [0,\deg(P_1)-1]$, from (6), we have $\det(M_i(P_0, P_1)) = \mathrm{lc}(\mathrm{Sres}_i(P_0, P_1, t))$ for all $i \in [0, \deg(P_1)-1]$, and hence, $\deg(\mathrm{Sres}_i(P_0,P_1,t)) = i$ for all $i \in [0, \deg(P_1)-1]$.

Assume that there exists some $k \in [1,q-1]$ s.t. $\deg(P_{k+1}) < \deg(P_k)-1$. Then, from (a), we have $\deg(\mathrm{Sres}_{\deg(P_k)-1}(P_0,P_1,t)) = \deg(P_{k+1}) < \deg(P_k)-1$, which contradicts $\deg(\mathrm{Sres}_i(P_0, P_1,t)) = i$ for all $i \in [0,\deg(P_1)-1]$. Therefore, we have $\deg(P_{k+1}) = \deg(P_k)-1 = \deg(P_1)-k$ for all $k \in [1,q-1]$.

Since $\deg(P_{k+1}) = \deg(P_k)-1 = \deg(P_1)-k$ for all $k \in [1,q-1]$ ensures $\deg(P_k)-\deg(P_{k+1})+1 = 2$ and $\deg(P_k)-\deg(P_{k+2}) = 2$ for all $k \in [1,q-2]$, we have

$$
\begin{aligned}
\lambda_{\deg(P_{k+1})} &= (-1)^{(\deg(P_0)-\deg(P_1)+1)(\deg(P_1)-\deg(P_{k+1}))}(\mathrm{lc}(P_1))^{\deg(P_0)-\deg(P_2)}\\
&\quad \times \prod_{n=1}^{k-2}(-1)^{2(\deg(P_{n+1})-\deg(P_{k+1}))}(\mathrm{lc}(P_{n+1}))^2\\
&\quad \times (-1)^2(\mathrm{lc}(P_k))^2\\
&= \Big((-1)^k \mathrm{lc}(P_1)\Big)^{\deg(P_0)-\deg(P_1)+1}\prod_{n=2}^{k}(\mathrm{lc}(P_n))^2.
\end{aligned}
$$

□

Appendix 7: Proof of Proposition 5

(A) Proof of (a): We derive computable expressions for

$$\deg(P_{l+1}), \mathrm{lc}(P_{l+1}) \text{ and } \mathrm{sgn}(\mathrm{lc}(P_{l+1})).$$

(I) Computable expression for $\deg(P_{l+1})$:

From Proposition 4(a), for $i \in [\deg(P_{l+1}), \deg(P_l) - 1]$, $\mathrm{Sres}_i(P_0, P_1, t)$ can be expressed as

$$\mathrm{Sres}_i(P_0,P_1,t) = \begin{cases} \lambda_{\deg(P_l)-1} P_{l+1}(t) \\ \qquad \text{for } i = \deg(P_l)-1, \\ \\ 0 \qquad \text{for } i \in [\deg(P_{l+1})+1, \deg(P_l)-2] \\ \qquad \text{(if } \deg(P_{l+1}) < \deg(P_l)-2\text{)}, \\ \\ \lambda_{\deg(P_{l+1})} (\mathrm{lc}(P_{l+1}))^{\deg(P_l)-\deg(P_{l+1})-1} P_{l+1}(t) \\ \qquad \text{for } i = \deg(P_{l+1}). \end{cases} \tag{29}$$

(i) If $\deg(P_{l+1}) = \deg(P_l) - 1$, we have, from the third expression in (29), $\deg(\mathrm{Sres}_{\deg(P_l)-1}(P_0, P_1, t)) = \deg(\mathrm{Sres}_{\deg(P_{l+1})}(P_0, P_1, t)) = \deg(P_{l+1}) = \deg(P_l) - 1$. Moreover, from (6), we have $\det(M_{\deg(P_l)-1}(P_0, P_1)) = \mathrm{lc}(\mathrm{Sres}_{\deg(P_{l+1})}(P_0, P_1, t)) \neq 0$. Hence, we have

$$\deg(P_{l+1}) = \deg(P_l) - \min\{s \in \mathbb{N}^* \mid \det(M_{\deg(P_l)-s}(P_0, P_1)) \neq 0\}.$$

(ii) If $\deg(P_{l+1}) = \deg(P_l) - 2$, let us examine first the case $s = 1$. Then we have, from the first expression in (29), $\deg(\mathrm{Sres}_{\deg(P_l)-1}(P_0, P_1, t)) = \deg(P_{l+1}) < \deg(P_l) - 1$. Moreover, from (6), we have $\det(M_{\deg(P_l)-1}(P_0, P_1)) = 0$.

Next, let us examine the case $s = 2$. Then we have, from the third expression in (29), $\deg(\mathrm{Sres}_{\deg(P_l)-2}(P_0, P_1, t)) = \deg(\mathrm{Sres}_{\deg(P_{l+1})}(P_0, P_1, t)) = \deg(P_{l+1}) = \deg(P_l) - 2$. Moreover, from (6), we have $\det(M_{\deg(P_l)-2}(P_0, P_1)) = \mathrm{lc}(\mathrm{Sres}_{\deg(P_{l+1})}(P_0, P_1, t)) \neq 0$. Hence, we have

$$\deg(P_{l+1}) = \deg(P_l) - \min\{s \in \mathbb{N}^* \mid \det(M_{\deg(P_l)-s}(P_0, P_1)) \neq 0\}.$$

(iii) If $\deg(P_{l+1}) \leq \deg(P_l) - 3$, let us examine first the case $s = 1$. Then we have, from the first expression in (29), $\deg(\mathrm{Sres}_{\deg(P_l)-1}(P_0, P_1, t)) = \deg(P_{l+1}) < \deg(P_l) - 1$. Moreover, from (6), we have $\det(M_{\deg(P_l)-1}(P_0, P_1)) = 0$.

Next let us examine the cases $s = \{2, 3, \ldots, \deg(P_l) - \deg(P_{l+1}) - 1\}$, we have, from the second expression in (29), $\deg(\mathrm{Sres}_{\deg(P_l)-s}(P_0, P_1, t)) = \deg(0) = -\infty < \deg(P_l) - s$. Moreover, from (6), we have $\det(M_{\deg(P_l)-2}(P_0, P_1)) = 0$.

Third let us examine the case $s = \deg(P_l) - \deg(P_{l+1})$, we have, from the third expression in (29), $\deg(\mathrm{Sres}_{\deg(P_l)-s}(P_0, P_1, t)) = \deg(\mathrm{Sres}_{\deg(P_{l+1})}(P_0, P_1, t)) = \deg(P_{l+1}) = \deg(P_l) - s$. Moreover, from (6), we have $\det(M_{\deg(P_l)-s}(P_0, P_1)) = \mathrm{lc}(\mathrm{Sres}_{\deg(P_{l+1})}(P_0, P_1, t)) \neq 0$. Hence, we have

$$\deg(P_{l+1}) = \deg(P_l) - \min\{s \in \mathbb{N}^* \mid \det(M_{\deg(P_l)-s}(P_0, P_1)) \neq 0\}.$$

(II) Computable expression for $\mathrm{lc}(P_{l+1})$ and $\mathrm{sgn}(\mathrm{lc}(P_{l+1}))$:

(i) If $(\deg(P_l)-\deg(P_{l+1}))$ is odd, from $\deg(\mathrm{Sres}_{\deg(P_{l+1})}(P_0,P_1,t))=\deg(P_{l+1})$, we have, from (6)

$$\begin{aligned}\det(M_{\deg(P_{l+1})}(P_0,P_1)) &= \mathrm{lc}(\mathrm{Sres}_{\deg(P_{l+1})}(P_0,P_1,t))\\ &= \lambda_{\deg(P_{l+1})}(\mathrm{lc}(P_{l+1}))^{\deg(P_l)-\deg(P_{l+1})-1}\times \mathrm{lc}(P_{l+1})\\ &= \lambda_{\deg(P_{l+1})}(\mathrm{lc}(P_{l+1}))^{\deg(P_l)-\deg(P_{l+1})}. \end{aligned}\tag{30}$$

Hence, we deduce

$$\mathrm{lc}(P_{l+1})=\sqrt[\deg(P_l)-\deg(P_{l+1})]{\frac{\det(M_{\deg(P_{l+1})}(P_0,P_1))}{\lambda_{\deg(P_{l+1})}}}.$$

Moreover, from (30), we have

$$\begin{aligned}&\mathrm{sgn}\left(\lambda_{\deg(P_{l+1})}\det(M_{\deg(P_{l+1})}(P_0,P_1))\right)\\ &=\mathrm{sgn}\left(\lambda^2_{\deg(P_{l+1})}(\mathrm{lc}(P_{l+1}))^{\deg(P_l)-\deg(P_{l+1})}\right)\\ &=\mathrm{sgn}\left((\mathrm{lc}(P_{l+1}))^{\deg(P_l)-\deg(P_{l+1})}\right)=\mathrm{sgn}(\mathrm{lc}(P_{l+1})).\end{aligned}$$

(ii) If $(\deg(P_l)-\deg(P_{l+1}))$ is even, i.e., $(\deg(P_l)-\deg(P_{l+1})-1)$ is odd, we have, from the first and third expressions of (29), for any $\tau\in\mathbb{R}$,

$$\left.\begin{aligned}\mathrm{Sres}_{\deg(P_l)-1}(P_0,P_1,\tau)=\lambda_{\deg(P_l)-1}P_{l+1}(\tau)\\ \mathrm{Sres}_{\deg(P_{l+1})}(P_0,P_1,\tau)=\lambda_{\deg(P_{l+1})}(\mathrm{lc}(P_{l+1}))^{\deg(P_l)-\deg(P_{l+1})-1}P_{l+1}(\tau)\end{aligned}\right\}.\tag{31}$$

Therefore, by using any $\tau\in\mathbb{R}$, we deduce

$$\mathrm{lc}(P_{l+1})=\sqrt[\deg(P_l)-\deg(P_{l+1})-1]{\frac{\lambda_{\deg(P_l)-1}\mathrm{Sres}_{\deg(P_{l+1})}(P_0,P_1,\tau)}{\lambda_{\deg(P_{l+1})}\mathrm{Sres}_{\deg(P_l)-1}(P_0,P_1,\tau)}}.$$

Moreover, from (31), we have

$$\begin{aligned}&\mathrm{sgn}\left(\lambda_{\deg(P_l)-1}\lambda_{\deg(P_{l+1})}\mathrm{Sres}_{\deg(P_l)-1}(P_0,P_1,\tau)\mathrm{Sres}_{\deg(P_{l+1})}(P_0,P_1,\tau)\right)\\ &=\mathrm{sgn}\left(\lambda^2_{\deg(P_l)-1}\lambda^2_{\deg(P_{l+1})}(\mathrm{lc}(P_{l+1}))^{\deg(P_l)-\deg(P_{l+1})-1}(P_{l+1}(\tau))^2\right)\\ &=\mathrm{sgn}\left((\mathrm{lc}(P_{l+1}))^{\deg(P_l)-\deg(P_{l+1})-1}\right)=\mathrm{sgn}(\mathrm{lc}(P_{l+1})).\end{aligned}$$

(B) Proof of (b): If $\det(M_i(P_0,P_1))\neq 0$ for all $i\in[0,\deg(P_1)-1]$, we can regard Eq. (12) as a special case of (A)-(II)-(i), and hence obtain, for all $k\in[1,q-1]$,

$$\mathrm{lc}(P_{k+1})=\frac{\det(M_{\deg(P_1)-k}(P_0,P_1))}{\left((-1)^k\mathrm{lc}(P_1)\right)^{\deg(P_0)-\deg(P_1)+1}\prod_{n=2}^{k}(\mathrm{lc}(P_n))^2}.$$

Moreover, since Eq. (12) ensures $\mathrm{sgn}(\lambda_{\deg(P_{k+1})}) = \mathrm{sgn}\Big(\big((-1)^k \mathrm{lc}(P_1)\big)^{(\deg(P_0)-\deg(P_1)+1)}\Big)$, we have, for all $k \in [1, q-1]$,

$$\begin{aligned}
\mathrm{sgn}(\mathrm{lc}(P_{k+1})) &= \mathrm{sgn}\left(\lambda_{\deg(P_{k+1})} \det(M_{\deg(P_{k+1})}(P_0,P_1))\right) \\
&= \mathrm{sgn}\left(\left((-1)^k \mathrm{lc}(P_1)\right)^{\deg(P_0)-\deg(P_1)+1} \det(M_{\deg(P_1)-k}(P_0,P_1))\right).
\end{aligned}$$

□

Appendix 8: Proof of Theorem 3

(A) Proof of (a): If $\deg(\Psi_0) \geq \deg(\Psi_1)$ and $q \geq 2$, from Proposition 4(a) and (8), we have

$$\begin{aligned}
\mathrm{sgn}(\Psi_k(t^*)) &= (-1)^{\frac{(k-1)k}{2}} \mathrm{sgn}(P_k(t^*)) \\
&= (-1)^{\frac{(k-1)k}{2}} \mathrm{sgn}\left(\frac{\mathrm{Sres}_{\deg(P_k)}(P_0,P_1,t^*)}{\lambda_{\deg(P_k)}(\mathrm{lc}(P_k))^{\deg(P_{k-1})-\deg(P_k)-1}}\right) \\
&= (-1)^{\frac{(k-1)k}{2}} \mathrm{sgn}\left(\frac{\mathrm{Sres}_{\deg(P_k)}(P_0,P_1,t^*)}{\lambda_{\deg(P_k)}(\mathrm{lc}(P_k))^{\deg(P_{k-1})-\deg(P_k)-1}}\right) \\
&\quad \times \mathrm{sgn}\left(\lambda^2_{\deg(P_k)}(\mathrm{lc}(P_k))^{2(\deg(P_{k-1})-\deg(P_k)-1)}\right) \\
&= (-1)^{\frac{(k-1)k}{2}} \mathrm{sgn}(\lambda_{\deg(P_k)}) \left(\mathrm{sgn}(\mathrm{lc}(P_k))\right)^{\deg(P_{k-1})-\deg(P_k)-1} \\
&\quad \times \mathrm{sgn}\left(\mathrm{Sres}_{\deg(P_k)}(P_0,P_1,t^*)\right) \\
&= (-1)^{\frac{(k-1)k}{2}} \kappa^{\langle 0\rangle}_{\deg(\Psi_k)} \left(\mathrm{sgn}(\mathrm{lc}(P_k))\right)^{\deg(\Psi_{k-1})-\deg(\Psi_k)-1} \\
&\quad \times \mathrm{sgn}\left(\mathrm{Sres}_{\deg(\Psi_k)}(\Psi_0,\Psi_1,t^*)\right),
\end{aligned}$$

where we used $\Psi_0(t) = P_0(t)$, $\Psi_1(t) = P_1(t)$, $\deg(\Psi_k) = \deg(P_k)$ and $\mathrm{sgn}(\lambda_{\deg(P_k)}) = \kappa^{\langle 0\rangle}_{\deg(\Psi_k)}$ (by (11)).

In particular, if $\det(M_i(\Psi_0,\Psi_1)) \neq 0$ for all $i \in [0, \deg(\Psi_1)-1]$, we have, from (12), $\deg(\Psi_q) = \deg(\Psi_1) - (q-1) = 0$, and hence $q = \deg(\Psi_1)+1$. Moreover we have, from $\deg(\Psi_{k-1}) - \deg(\Psi_k) - 1 = 0$ and (12), $\kappa^{\langle 0\rangle}_{\deg(\Psi_k)} \left(\mathrm{sgn}(\mathrm{lc}(P_k))\right)^{\deg(\Psi_{k-1})-\deg(\Psi_k)-1} = \kappa^{\langle 0\rangle}_{\deg(\Psi_k)} = \mathrm{sgn}(\lambda_{\deg(P_k)}) = (-1)^{(k-1)(\deg(\Psi_0)-\deg(\Psi_1)-1)} \left(\mathrm{sgn}(\mathrm{lc}(\Psi_1))\right)^{\deg(\Psi_0)-\deg(\Psi_1)+1}$. As a result, we have (14).

(B) Proof of (b): If $\deg(\Psi_0) < \deg(\Psi_1)$, i.e., $\deg(P_0) < \deg(P_1)$, and $q \geq 3$, $P_2(t) = P_0(t) - 0 \times P_1(t) = P_0(t)$ and we have $\deg(P_1) > \deg(P_2) > \cdots > \deg(P_q)$. Then by replacing $P_0(t)$ and $P_1(t)$ in Proposition 4(a) with $P_1(t)$ and $P_2(t)$, for any $k = \{2,3,\ldots,q-1\}$, $\mathrm{Sres}_i(P_1,P_0,t)$ ($i \in [\deg(P_{k+1}), \deg(P_k)-1]$) can be expressed as

$$\mathrm{Sres}_i(P_1,P_0,t) = \mathrm{Sres}_i(P_1,P_2,t) = \begin{cases} \lambda^{\langle 1\rangle}_{\deg(P_k)-1} P_{k+1}(t) \\ \quad \text{for } i = \deg(P_k)-1, \\ 0 \quad \text{for } i \in [\deg(P_{k+1})+1, \deg(P_k)-2] \\ \quad (\text{if } \deg(P_{k+1}) < \deg(P_k)-2), \\ \lambda^{\langle 1\rangle}_{\deg(P_{k+1})} (\mathrm{lc}(P_{k+1}))^{\deg(P_k)-\deg(P_{k+1})-1} P_{k+1}(t) \\ \quad \text{for } i = \deg(P_{k+1}), \end{cases}$$

where, for $i = \deg(P_k) - 1, \deg(P_{k+1})$,

$$\lambda_i^{\langle 1 \rangle} := \prod_{n=1}^{k-2} (-1)^{(\deg(P_n) - \deg(P_{n+1}) + 1)(\deg(P_{n+1}) - i)} (\mathrm{lc}(P_{n+1}))^{\deg(P_n) - \deg(P_{n+2})}$$
$$\times (-1)^{(\deg(P_{k-1}) - \deg(P_k) + 1)(\deg(P_k) - i)} (\mathrm{lc}(P_k))^{\deg(P_{k-1}) - i}.$$

As a result, we have,

$$\begin{aligned}
\mathrm{sgn}(\Psi_k(t^*)) &= (-1)^{\frac{(k-1)k}{2}} \mathrm{sgn}(P_k(t^*)) \\
&= (-1)^{\frac{(k-1)k}{2}} \mathrm{sgn}\left(\frac{\mathrm{Sres}_{\deg(P_k)}(P_1, P_0, t^*)}{\lambda_{\deg(P_k)}^{\langle 1 \rangle} (\mathrm{lc}(P_k))^{\deg(P_{k-1}) - \deg(P_k) - 1}} \right) \\
&= (-1)^{\frac{(k-1)k}{2}} \mathrm{sgn}\left(\frac{\mathrm{Sres}_{\deg(P_k)}(P_1, P_0, t^*)}{\lambda_{\deg(P_k)}^{\langle 1 \rangle} (\mathrm{lc}(P_k))^{\deg(P_{k-1}) - \deg(P_k) - 1}} \right) \\
&\quad \times \mathrm{sgn}\left(\left(\lambda_{\deg(P_k)}^{\langle 1 \rangle} \right)^2 (\mathrm{lc}(P_k))^{2(\deg(P_{k-1}) - \deg(P_k) - 1)} \right) \\
&= (-1)^{\frac{(k-1)k}{2}} \mathrm{sgn}(\lambda_{\deg(P_k)}^{\langle 1 \rangle}) \big(\mathrm{sgn}(\mathrm{lc}(P_k)) \big)^{\deg(P_{k-1}) - \deg(P_k) - 1} \\
&\quad \times \mathrm{sgn}\big(\mathrm{Sres}_{\deg(P_k)}(P_1, P_0, t^*) \big) \\
&= (-1)^{\frac{(k-1)k}{2}} \kappa_{\deg(\Psi_k)}^{\langle 1 \rangle} \big(\mathrm{sgn}(\mathrm{lc}(P_k)) \big)^{\deg(\Psi_{k-1}) - \deg(\Psi_k) - 1} \\
&\quad \times \mathrm{sgn}\big(\mathrm{Sres}_{\deg(\Psi_k)}(\Psi_1, \Psi_0, t^*) \big),
\end{aligned}$$

where we used $\Psi_0(t) = P_0(t)$, $\Psi_1(t) = P_1(t)$, $\deg(\Psi_k) = \deg(P_k)$ and $\mathrm{sgn}(\lambda_{\deg(P_k)}^{\langle 1 \rangle}) = \kappa_{\deg(\Psi_k)}^{\langle 1 \rangle}$.

In particular, if $\det(M_i(\Psi_1, \Psi_0)) \neq 0$ for all $i \in [0, \deg(\Psi_0) - 1]$, in almost same way as in proof for (A), we have $q = \deg(\Psi_0) + 2$ and (16). $\square$

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