ALGEBRAIC PHASE UNWRAPPING OVER COLLECTION OF TRIANGLES BASED ON TWO-DIMENSIONAL SPLINE SMOOTHING

Daichi Kitahara and Isao Yamada

Department of Communications and Computer Engineering, Tokyo Institute of Technology, Japan
E-mail: {kitahara, isao}@sp.ce.titech.ac.jp

ABSTRACT

Phase unwrapping is a reconstruction problem of the continuous phase function from its finite wrapped samples. Especially the two-dimensional phase unwrapping has been a common key for estimating many crucial physical information, e.g., the surface topography measured by interferometric synthetic aperture radar. However almost all two-dimensional phase unwrapping algorithms are suffering from either the path dependence or the excess smoothness of the estimated result. In this paper, to guarantee the path independence and the appropriate smoothness of the estimated result, we present a novel algebraic approach by combining the ideas in the algebraic phase unwrapping with techniques for a piecewise polynomial interpolation of two-dimensional finite data sequence.

Index Terms— Algebraic phase unwrapping, Functional data analysis, Two-dimensional phase unwrapping, Spline smoothing, Interferometric synthetic aperture radar

1. INTRODUCTION

Two-dimensional (2D) phase unwrapping [1] is a reconstruction problem of the unknown unwrapped phase $\Theta(x,y) \in \mathbb{R}$ defined in a simply connected region $\Omega \subset \mathbb{R}^2$, from its finite wrapped samples $[\Theta(x,y)]_{\mathrm{mod}\ 2\pi} \in (-\pi,\pi]$ observed at $(x,y) \in \mathcal{G}(\subset \Omega)$, where $\mathcal{G}$ stands for the set of finite grid points. The 2D phase unwrapping has been a common key for estimating many crucial physical information such as the surface topography measured by interferometric synthetic aperture radar (InSAR) [2, 3, 4] or interferometric synthetic aperture sonar (InSAS) [5], the degree of magnetic field in homogeneity in the water/fat separation problem in magnetic resonance imaging (MRI) [6], and the accurate profile of mechanical parts by x-ray [7].

Major existing 2D phase unwrapping algorithms, e.g., path-following methods [2, 8, 9, 10] and network flow methods [4, 11, 12], estimate the unwrapped phase as $\widetilde{\Theta}(x,y) := [\Theta(x,y)]_{\mathrm{mod}\ 2\pi} + 2\pi\eta(x,y)$ with $\eta : \mathcal{G} \to \mathbb{Z}$, by trying to find η^* which minimizes the cardinality of

$$\left\{ (x,y) \;\middle|\; \exists (x',y') \in \mathcal{N}(x,y) \;\; \left|\widetilde{\Theta}(x,y) - \widetilde{\Theta}(x',y')\right| > \pi \right\}, \quad (1)$$

where $\mathcal{N}(x,y)$ is the set of neighboring grid points of (x,y). Unfortunately, this combinatorial problem is intractable due to its NP-Hardness [4]. As a result, such algorithms are suffering from the so-called *path dependence* of the estimated unwrapped phase, i.e., the estimated result differs depending on the execution procedure of the algorithm. The other algorithms, e.g., minimum-norm methods [13, 14, 15] which find $\widetilde{\Theta}^*$ minimizing

$$\sum_{(x,y)\in\mathcal{G}} \sum_{(x',y')\in\mathcal{N}(x,y)} \left|\widetilde{\Theta}(x,y) - \widetilde{\Theta}(x',y') - \delta_{(x',y')}^{(x,y)}\right|^p, \quad (2)$$

where $\delta_{(x',y')}^{(x,y)} := \Big[[\Theta(x,y)]_{\mathrm{mod}\ 2\pi} - [\Theta(x',y')]_{\mathrm{mod}\ 2\pi}\Big]_{\mathrm{mod}\ 2\pi}$ and $p \geq 1$, are likely to make the estimate of the unwrapped phase too smooth, which conflicts with observed data even though the path independence is guaranteed. These situations imply that any technically reliable solution has not yet been established even though the failure of the 2D phase unwrapping makes a substantial impact on the accuracy of the estimated physical information.

In this paper, we propose a completely different algebraic approach to the 2D phase unwrapping problem. We estimate Θ by θ_f, where θ_f is the unwrapped phase of a twice differentiable complex function $f := f_{(0)} + jf_{(1)}$ s.t. $f_{(i)} : \mathbb{R}^2 \to \mathbb{R}$ $(i = 0, 1)$. Then the estimation problem of Θ is replaced with that of f, i.e., $(f_{(0)}, f_{(1)})$. The proposed approach is designed based on Theorem 1 in APPENDIX A, which was established recently in [16, 17], to guarantee the unique existence of $\theta_f \in C^2(\Omega)$ as a scalar potential having, as its gradient flow, the partial derivatives of the wrapped phase function. In the spirit of functional data analysis [18, 19]: *"smoothness of estimate should be measured for functions which possibly generate the data,"* we use best smoothing functions $f_{(0)}^*$ and $f_{(1)}^*$ among all possible candidates, in a suitable functional space, which are consistent with given wrapped phase information. As a result, we obtain a best scalar potential θ_f^* as an estimate of the unwrapped phase surface Θ.

The proposed algorithm is realized by combining the ideas in the algebraic phase unwrapping [16, 17, 20, 21, 22] with so-called *spline smoothing* which constructs a piecewise polynomial interpolation [23, 24, 25, 26] of two-dimensional finite data sequence, as a best solution of a variational problem. Remarkably, unlike almost all existing algorithms, the proposed algorithm achieves not only the path independence but also the appropriate smoothness of the estimated unwrapped phase under a reasonable assumption. Numerical experiments, based on the topographic mapping by InSAR, demonstrate the effectiveness of the proposed 2D phase unwrapping.

Relation to Prior Work

The work presented here focuses on the 2D phase unwrapping [1]. We propose a novel 2D phase unwrapping algorithm based on the algebraic phase unwrapping [20, 21, 22] and the functional data analysis [18, 19]. In our previous papers [16, 17], we elucidated the condition for the path independence of the 2D phase unwrapping, and extended the algebraic phase unwrapping to the 2D phase unwrapping for a pair of bivariate polynomials. In this paper, in the spirit of functional data analysis, we propose to choose, as an optimal preprocessing of the extended algebraic phase unwrapping, the pair of bivariate interpolating spline functions which are the best smoothing functions in a suitable spline space.

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2. PRELIMINARIES

2.1. Notation and Bivariate Spline Space

Let $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{R}$, and $\mathbb{C}$ denote respectively the set of all integers, nonnegative integers, real numbers, and complex numbers. We use $j \in \mathbb{C}$ to denote the imaginary unit satisfying $j^2 = -1$. For any $c \in \mathbb{C}$, $\Im(c)$ stands for the imaginary part of c.

Let Δ be a collection of triangles whose union forms a simply connected region $\Omega \subset \mathbb{R}^2$. For any two triangles $\mathcal{T}, \mathcal{T}' \in \Delta$, if $\mathcal{T} \cap \mathcal{T}'$ is either empty or a common edge of $\mathcal{T}$ and $\mathcal{T}'$ or a common vertex of $\mathcal{T}$ and $\mathcal{T}'$, Δ is called a *regular triangulation*.

Given two integers $d \geq 0$ and $0 \leq \rho < d$, define

$$\mathcal{S}_d^\rho(\Delta) := \{f \in C^\rho(\Omega) \mid \text{for all } \mathcal{T} \in \Delta,\ f = f_\mathcal{T} \in \mathbb{P}_d \text{ over } \mathcal{T}\}$$

as the set of all bivariate spline functions of degree d and smoothness ρ, where $C^\rho(\Omega)$ stands for the set of all ρ-times continuously differentiable real valued functions over Ω, and $\mathbb{P}_d$ denotes the set of all polynomials whose degree is d at most.

2.2. B-form Representation of Spline Functions

Let $\mathcal{T} = \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle \in \Delta$ be a triangle in Δ, i.e., $\boldsymbol{v}_k = (x_k, y_k)$ $(k = 1, 2, 3)$ are not arranged linearly, and hence every point $(x, y) \in \mathbb{R}^2$ can be expressed uniquely in the form

$$(x, y) = r\boldsymbol{v}_1 + s\boldsymbol{v}_2 + t\boldsymbol{v}_3 \qquad \text{s.t. } r + s + t = 1,$$

where (r, s, t) are called the *barycentric coordinates* of the point (x, y) with respect to the triangle $\mathcal{T}$.

Remark 1 *For all $(x, y) \in \mathcal{T}$, the barycentric coordinates (r, s, t) are expressed respectively as*

$$\left.\begin{array}{rcl} r &=& \{(y_2 - y_3)x - (x_2 - x_3)y + x_2y_3 - y_2x_3\}/\kappa \\ s &=& \{(y_3 - y_1)x - (x_3 - x_1)y + x_3y_1 - y_3x_1\}/\kappa \\ t &=& \{(y_1 - y_2)x - (x_1 - x_2)y + x_1y_2 - y_1x_2\}/\kappa \end{array}\right\},$$

where $\kappa := x_1y_2 - y_1x_2 + x_2y_3 - y_2x_3 + x_3y_1 - y_3x_1$ [24].

For integers $l \geq 0$, $m \geq 0$ and $n \geq 0$, the *Bernstein-Bézier polynomial* is defined as

$$B_{l,m,n}^d(r, s, t) := \frac{d!}{l!m!n!} r^l s^m t^n \qquad \text{s.t. } l + m + n = d.$$

For a fixed d, $\{B_{l,m,n}^d \in \mathbb{P}_d \mid l + m + n = d \text{ and } l, m, n \in \mathbb{Z}_+\}$ is a basis of the space of polynomials corresponding to $\mathbb{P}_d$. As a result, any spline function $f \in \mathcal{S}_d^\rho(\Delta)$ restricted to each triangle $\mathcal{T} \in \Delta$ can be written uniquely as

$$f_\mathcal{T}(r, s, t) = \sum_{l+m+n=d} c_{l,m,n}^\mathcal{T} B_{l,m,n}^d(r, s, t),$$

where $c_{l,m,n}^\mathcal{T} \in \mathbb{R}$. Such a representation is called the *B-form representation* of the spline function f. We denote the B-coefficient vector of f by

$$\boldsymbol{c} := \{c_{l,m,n}^\mathcal{T} \mid \mathcal{T} \in \Delta,\ l + m + n = d, \text{ and } l, m, n \in \mathbb{Z}_+\}.$$

Example 1 *Let us consider a simple example, where the triangulation Δ has only one triangle $\mathcal{T} = \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle = \Omega$, i.e., $\Delta := \{\mathcal{T}\}$. Suppose that the degree d of a spline function f is 4, and the B-coefficient vector $\boldsymbol{c} := (c_1, c_2, \ldots, c_{15})^T$ is denoted as shown in Fig. 1. Then the spline function $f = f_\mathcal{T}$ is expressed, in terms of the barycentric coordinates (r, s, t), by Eq. (3).*

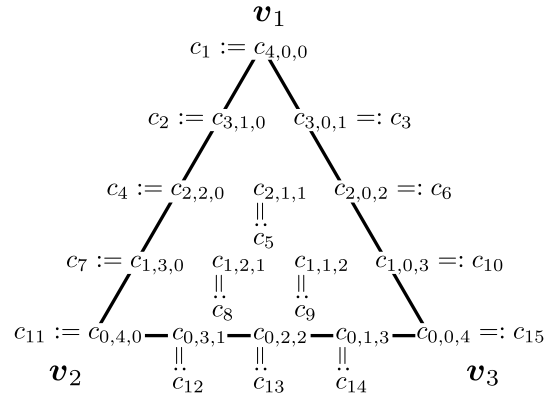


Fig. 1. Triangle $\mathcal{T} = \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle$ and B-coefficients for degree 4

3. SMOOTH PHASE SURFACE AS A SCALAR POTENTIAL

3.1. Proposed Method

In our previous works [16, 17], we clarified the condition of the unique existence of the 2D unwrapped phase $\theta_f \in C^2(\Omega)$ of a twice differentiable complex function $f := f_{(0)} + jf_{(1)}$ s.t. $f_{(i)} : \mathbb{R}^2 \to \mathbb{R}$ $(i = 0, 1)$ (see Theorem 1 in APPENDIX A).

Trying to estimate Θ by θ_f, from Theorem 1, we can reduce the estimation problem of Θ to that of $f_{(i)} \in C^2(\Omega)$ $(i = 0, 1)$ satisfying $f(x, y) = f_{(0)}(x, y) + jf_{(1)}(x, y) \neq 0$ for all $(x, y) \in \Omega$. Hence we can design θ_f, as an ideal estimate of Θ in the sense of the functional data analysis [18, 19], by suppressing optimally the rapid local changes of $f_{(0)} : (x, y) \mapsto a(x, y)\cos(\theta_f(x, y))$ and $f_{(1)} : (x, y) \mapsto a(x, y)\sin(\theta_f(x, y))$ in $\mathcal{S}_d^\rho(\Delta)$ $(d > \rho \geq 2)$, subject to $a(x, y) > 0$ $(\forall (x, y) \in \Omega)$ and

$$\left.\begin{array}{l} f_{(0)}(x, y) = \cos([\Theta(x, y)]_{\text{mod } 2\pi}) \\ f_{(1)}(x, y) = \sin([\Theta(x, y)]_{\text{mod } 2\pi}) \end{array}\right\} \text{ for all } (x, y) \in \mathcal{G}.$$

As a result, we consider the 2D phase unwrapping as the following problem which consists of two steps.

Problem 1

Step 1 *Find $f_{(i)}^* \in \mathcal{S}_d^\rho(\Delta) \subset C^\rho(\Omega)$ $(i = 0, 1)$ which minimizes*

$$J(f_{(i)}) := \iint_\Omega \left[\left| \frac{\partial^2 f_{(i)}}{\partial x^2} \right|^2 + 2 \left| \frac{\partial^2 f_{(i)}}{\partial x \partial y} \right|^2 + \left| \frac{\partial^2 f_{(i)}}{\partial y^2} \right|^2 \right] dxdy \quad (4)$$

subject to

$$\left.\begin{array}{l} f_{(0)}(x, y) = \cos([\Theta(x, y)]_{\text{mod } 2\pi}) \\ f_{(1)}(x, y) = \sin([\Theta(x, y)]_{\text{mod } 2\pi}) \end{array}\right\} \textit{ for all } (x, y) \in \mathcal{G}.$$

Step 2 *For any point of interest $(x, y) \in \Omega$, compute $\theta_{f^*}(x, y)$ defined in Theorem 1 by along a suitable piecewise C^1 path γ (see Eq. (6) in APPENDIX A).*

Remark 2 *Problem 1 is a convex relaxation of an ideal optimization problem which includes an additional constraint $f_{(0)} + jf_{(1)} \neq 0$ over Ω. Fortunately, if sufficiently many grid points are employed for Θ, the solution $(f_{(0)}^*, f_{(1)}^*)$ of this relaxed problem tends to satisfy automatically $f_{(0)}^* + jf_{(1)}^* \neq 0$ over Ω because the sum of squares achieves 1 at every grid points and the rapid local change $J(f_{(i)}^*)$ $(i = 0, 1)$ are suppressed globally.*

$$f_{\mathcal{T}}(r,s,t) = c_1r^4 + 4c_2r^3s + 4c_3r^3t + 6c_4r^2s^2 + 12c_5r^2st + 6c_6r^2t^2 + 4c_7rs^3 + 12c_8rs^2t + 12c_9rst^2 + 4c_{10}rt^3 + c_{11}s^4 + 4c_{12}s^3t + 6c_{13}s^2t^2 + 4c_{14}st^3 + c_{15}t^4 \quad (3)$$

3.2. Solution of Step 1 in Problem 1

As shown in [27], $J(f_{(i)})$ in (4) can be written as a quadratic form $J(f_{(i)}) = J(\boldsymbol{c}_{(i)}) = \boldsymbol{c}_{(i)}^T\boldsymbol{Q}\boldsymbol{c}_{(i)}$, where $\boldsymbol{Q}$ is a symmetric positive semi-definite matrix. Moreover, if the condition $f_{(i)} \in \mathcal{S}_d^{\rho}(\Delta)$ and the interpolating condition are feasible, these can be respectively written as $\boldsymbol{\mathcal{H}}\boldsymbol{c}_{(i)} = \boldsymbol{0}$ and $\boldsymbol{\mathcal{I}}\boldsymbol{c}_{(i)} = \boldsymbol{d}_{(i)}$, where $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{I}}$ are certain sparse matrices [24, 25, 26], and $\boldsymbol{d}_{(i)}$ $(i = 0, 1)$ are given by

$$\left.\begin{array}{l} \boldsymbol{d}_{(0)} := \{\, \cos([\Theta(x,y)]_{\bmod 2\pi}) \mid (x,y) \in \mathcal{G}\} \\ \boldsymbol{d}_{(1)} := \{\, \sin([\Theta(x,y)]_{\bmod 2\pi}) \mid (x,y) \in \mathcal{G}\} \end{array}\right\}.$$

Therefore, in this case, Step 1 in Problem 1 is replaced with the following convex optimization problem.

Step 1 *Find $\boldsymbol{c}_{(i)}^*$ $(i = 0, 1)$ which minimizes*

$$\boldsymbol{c}_{(i)}^T\boldsymbol{Q}\boldsymbol{c}_{(i)} \qquad \textit{s.t.}\ \ \boldsymbol{\mathcal{H}}\boldsymbol{c}_{(i)} = \boldsymbol{0}\ \textit{and}\ \boldsymbol{\mathcal{I}}\boldsymbol{c}_{(i)} = \boldsymbol{d}_{(i)}.$$

By considering the additive noise, we generalize the above optimization problem as a kind of the generalized Hermite-Birkhoff interpolation problem [18]:

Step 1′ *Find $\boldsymbol{c}_{(i)}^*(i = 0, 1)$ which minimizes*

$$\boldsymbol{c}_{(i)}^T\boldsymbol{Q}\boldsymbol{c}_{(i)} \qquad \textit{s.t.}\ \ \boldsymbol{\mathcal{H}}\boldsymbol{c}_{(i)} = \boldsymbol{0}\ \textit{and}\ -\epsilon\boldsymbol{1} \le \boldsymbol{\mathcal{I}}\boldsymbol{c}_{(i)} - \boldsymbol{d}_{(i)} \le \epsilon\boldsymbol{1},$$

where $\epsilon > 0$ and $\boldsymbol{1}$ denotes the vector whose all components is 1. Step 1 and Step 1′ are respectively solved by [28] and [29] if the above constraint sets are not empty.

Moreover, if we introduce the idea of the hierarchical convex optimization [30]:

Step 1″ *Find $\boldsymbol{c}_{(i)}^*(i = 0, 1)$ which minimizes*

$$\boldsymbol{c}_{(i)}^T\boldsymbol{Q}\boldsymbol{c}_{(i)} \qquad \textit{s.t.}\ \ \boldsymbol{c}_{(i)} \in \mathop{\arg\min}_{\boldsymbol{\mathcal{H}}\boldsymbol{c}_{(i)} = \boldsymbol{0}} \ ||\boldsymbol{\mathcal{I}}\boldsymbol{c}_{(i)} - \boldsymbol{d}_{(i)}||^2,$$

we can always achieve optimal spline function $f_{(i)}^* \in \mathcal{S}_d^{\rho}(\Delta)$ by minimizing the rapid local change while admitting minimal deviation from the sampling data $\boldsymbol{d}_{(i)}$. Such a hierarchical optimization problem can be solved by the *hybrid steepest descent method* [30, 31].

3.3. Solution of Step 2 in Problem 1

By the path independence guaranteed by Theorem 1 and the choice of bivariate splines for $(f_{(0)}^*, f_{(1)}^*)$, the line integral (6), can be decomposed into a finite sum of integrals of the following type:

$$\int_a^{t^*} \Im\left[\frac{A_{(0)}'(t) + jA_{(1)}'(t)}{A_{(0)}(t) + jA_{(1)}(t)}\right] dt,$$

where $A_{(0)}$ and $A_{(1)}$ are univariate real polynomials satisfying $A_{(0)}(t) + jA_{(1)}(t) \neq 0$ $(\forall t \in [a, t^*])$. Fortunately, a closed form expression of the above integral, for nontrivial cases: $A_{(0)} \not\equiv 0$ and $A_{(1)} \not\equiv 0$, is given by the *algebraic phase unwrapping* [16, 17, 20, 21, 22] (see Theorem 2 in APPENDIX B). Therefore we can compute the integral (6) for $(f_{(0)}^*, f_{(1)}^*)$ without requiring any knowledge on the location of zeros of $f_{(0)}^*$.

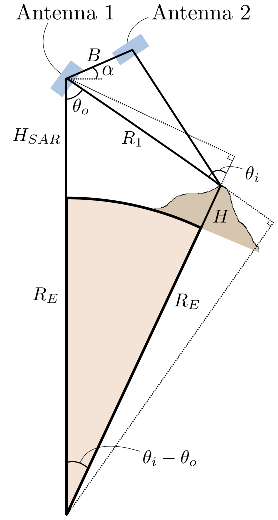


Fig. 2. Principle of topographic mapping by InSAR

Table 1. Parameters of InSAR

λ	2.35×10^{-1} [m]
H_{SAR}	1.0×10^{4} [m]
R_E	6.371×10^{6} [m]
B	1.3×10 [m]
α	0 [rad]

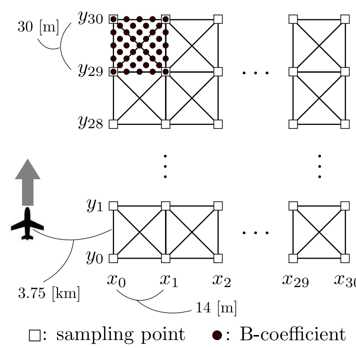


Fig. 3. Setting of $\mathcal{G}$ and $\Delta := \Delta_{\dagger}$

4. EXPERIMENTS BASED ON TOPOGRAPHIC MAPPING

4.1. Altitude Estimation by InSAR

The interferometric synthetic aperture radar (InSAR) [2, 3, 4] is an imaging technique allowing highly accurate measurements of a surface topography in all weather conditions, day or night. In the InSAR system, a pair of antennas, on-board an aircraft or a spacecraft platform, transmits coherent microwave radio signals and then receives the reflected signals from a common target at a position in a landsurface. Figure 2 shows the principle of topographic mapping and tells us that the altitude H of the target can be expressed as

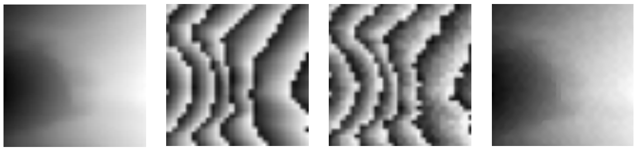
$$H = (H_{SAR} + R_E)\cos(\theta_i - \theta_o) - R_E - R_1\cos\theta_i, \quad (5)$$

where H_{SAR}, R_E, and R_1, which denote respectively the height of SAR, the radius of earth, and the distance from Antenna 1 to the target, are available, and the incidence angle θ_i is given, with use of the off-nadir angle θ_o by $\theta_i = \arctan\left(\frac{(H_{SAR}+R_E)\sin\theta_o}{(H_{SAR}+R_E)\cos\theta_o - R_1}\right)$.

To estimate the altitude $H(x,y)$ of the target at position $(x,y) \in \Omega$, we need the actual phase difference $\Theta(x,y)$ between two signals received by the antennas of the InSAR system because Θ is directly related to the off-nadir angle θ_o and the wavelength λ of the signal by $\Theta(x,y) = \frac{2\pi B\sin(\theta_o(x,y)-\alpha)}{\lambda}$, where B and α respectively denote the distance and the elevation angle from Antenna 1 to Antenna 2.

4.2. Numerical Experiments

In what follows, assume that the set of finite sampling points is a regular rectangular grid on the area $\Omega := [x_0, x_{30}] \times [y_0, y_{30}]$, i.e., $\mathcal{G} := \{(x_{k_1}, y_{k_2})\}_{k_1,k_2}$ to be given by $x_0 < x_1 < \ldots < x_{30}$ and $y_0 < y_1 < \ldots < y_{30}$ satisfying $x_{k_1+1} - x_{k_1} = 14$ $[m]$ and $y_{k_2+1} - y_{k_2} = 30$ $[m]$ $(k_1 = 0, 1, \ldots, 29$ and $k_2 = 0, 1, \ldots, 29)$. Moreover, we construct a triangulation $\Delta_{\dagger}$ by dividing every rectangular $[x_{k_1}, x_{k_1+1}] \times [y_{k_2}, y_{k_2+1}]$ into four triangles as illustrated in Fig. 3. This triangulation is called the *crisscross partition*. The other parameters are written in Table 1.



(a) Θ (b) $[\Theta]_{\text{mod } 2\pi}$ (c) $[\Theta+\nu]_{\text{mod } 2\pi}$ (d) Proposed

Fig. 4. Result of the proposed 2D phase unwrapping

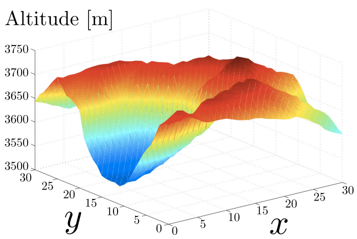


Fig. 5. Proposed estimate of altitude

Table 2. MSE of the estimate

Algorithm	MSE
Branch Cut	0.0923
Least Squares	0.1655
Proposed	**0.0850**

$$\text{MSE} := \frac{1}{|\mathcal{G}|}\sum_{(x,y)\in\mathcal{G}}\{\Theta(x,y)-\widetilde{\Theta}(x,y)\}^2$$

($\widetilde{\Theta}$: estimate of Θ)

We show the effectiveness of the proposed algorithm in the altitude estimation from noisy wrapped samples. Figures 4(a) and (b) depict respectively the unknown unwrapped phase Θ and its wrapped phase $[\Theta]_{\text{mod } 2\pi}$. We can observe only noisy wrapped samples $[\Theta(x,y)+\nu(x,y)]_{\text{mod } 2\pi}$ at $(x,y)\in\mathcal{G}$ as shown in Fig. 4(c), where ν is the additive noise generated by the uniform distribution between $-\pi/6$ and $\pi/6$. In this case, we use $\mathcal{S}_4^2(\Delta_\dagger)$ as the bivariate spline space, and adopt Step $1'$ with $\epsilon=0.2$ as the first step in the proposed algorithm. Figures 4(d) depicts the estimated result θ_{f^*} by the proposed algorithm and Fig. 5 shows the estimate, constructed by (5), of the altitude of the vicinity of the volcanic vent of Mt. Fuji. Table 2 shows the mean square errors (MSE) of three estimates computed by the branch cut algorithm [2], which is the algorithm whose strategy is Eq. (1), by the least squares algorithm [13], which is the algorithm whose strategy is Eq. (2) with $p=2$, and by the proposed algorithm. From Table 2, we observe that the proposed algorithm achieves the lowest mean square error.

5. CONCLUSION

In this paper, we have proposed a novel 2D phase unwrapping algorithm which is composed of 2 steps: *the convex optimization based on spline smoothing, and the recovery of the scalar potential function by the algebraic phase unwrapping.* Remarkably, unlike almost all existing algorithms, the proposed approach guarantees not only the path independence but also the appropriate smoothness of the unwrapped phase. By comparing the proposed algorithm with existing algorithms, the effectiveness of the proposed approach has been confirmed in numerical experiments based on topographic mapping.

APPENDIX A

The next theorem, derived by using Poincaré's lemma [32], motivated us to formulate the 2D phase unwrapping as Problem 1.

Theorem 1 (2D phase unwrapping as a scalar potential [16, 17]) *Suppose that $f_{(i)}:\mathbb{R}^2\to\mathbb{R}$ $(i=0,1)$ are $C^2(\Omega)$ functions satisfying $f(x,y):=f_{(0)}(x,y)+jf_{(1)}(x,y)\neq 0$ for all $(x,y)\in\Omega$. Then for an arbitrarily fixed $(x_0,y_0)\in\Omega$ and $\theta_0\in(-\pi,\pi]$ satisfying $f(x_0,y_0)=|f(x_0,y_0)|e^{j\theta_0}$, the following hold.*

Algorithm 1 Sturm generating algorithm along the real axis (Sturm-$\mathcal{R}$)

Input: $A_{(0)}(t), A_{(1)}(t)\in\mathbb{R}[t]$ and $a\in\mathbb{R}$ under the assumptions

$\Psi_0(t)\leftarrow\dfrac{A_{(0)}(t)}{(t-a)^{e_0}},\ \Psi_1(t)\leftarrow\dfrac{A_{(1)}(t)}{(t-a)^{e_1}}$
(e_i : the order of $t=a$ as a zero of polynomial $A_{(i)}(t)$ $(i=0,1)$)

$k\leftarrow 1$

while $\deg(\Psi_k)\neq 0$ **do**

  $\Psi_{k+1}(t)\leftarrow -\Psi_{k-1}(t)-H_k(t)\Psi_k(t)$
  (where $H_k(t)\in\mathbb{R}[t]$ and $\deg(\Psi_{k+1})<\deg(\Psi_k)$)

  $k\leftarrow k+1$

end while

$q\leftarrow\begin{cases} k & \text{if } \Psi_k(t)\not\equiv 0\\ k-1 & \text{if } \Psi_k(t)\equiv 0\end{cases}$

Output: $\{\Psi_k(t)\}_{k=0}^q$

(a) *There exists a unique $\theta_f\in C^2(\Omega)$ satisfying $\theta_f(x_0,y_0)=\theta_0$ and for all $(x,y)\in\Omega$*

$$\left.\begin{aligned}\frac{\partial\theta_f}{\partial x}(x,y)&=\Im\left[\frac{\frac{\partial f_{(0)}}{\partial x}(x,y)+j\frac{\partial f_{(1)}}{\partial x}(x,y)}{f_{(0)}(x,y)+jf_{(1)}(x,y)}\right]\\ \frac{\partial\theta_f}{\partial y}(x,y)&=\Im\left[\frac{\frac{\partial f_{(0)}}{\partial y}(x,y)+j\frac{\partial f_{(1)}}{\partial y}(x,y)}{f_{(0)}(x,y)+jf_{(1)}(x,y)}\right]\end{aligned}\right\}.$$

In other words, θ_f is a scalar potential of

$$\left(\Im\left[\frac{\frac{\partial f_{(0)}}{\partial x}(x,y)+j\frac{\partial f_{(1)}}{\partial x}(x,y)}{f_{(0)}(x,y)+jf_{(1)}(x,y)}\right],\Im\left[\frac{\frac{\partial f_{(0)}}{\partial y}(x,y)+j\frac{\partial f_{(1)}}{\partial y}(x,y)}{f_{(0)}(x,y)+jf_{(1)}(x,y)}\right]\right).$$

(b) *$\theta_f\in C^2(\Omega)$, defined in* (a)*, is given by*

$$\theta_f(x,y)=\theta_0+\int_0^1\Im\left[\frac{\left(f_{(0)}(\gamma(t))\right)'+j\left(f_{(1)}(\gamma(t))\right)'}{f_{(0)}(\gamma(t))+jf_{(1)}(\gamma(t))}\right]dt,\quad(6)$$

where $\gamma:[0,1]\to\Omega$ is any piecewise C^1 path satisfying $\gamma(0)=(x_0,y_0)$ and $\gamma(1)=(x,y)$.

APPENDIX B

The next theorem was derived by extending Sturm's genius use [33] of the Euclidean algorithm for extraction of the root location of a real polynomial.

Theorem 2 (Algebraic phase unwrapping [16, 17, 20, 21, 22]) *Let $A(t):=A_{(0)}(t)+jA_{(1)}(t)\in\mathbb{C}[t]$ satisfy $A(t)\neq 0$ $(t\in[a,b])$, where $A_{(0)}(t),A_{(1)}(t)\in\mathbb{R}[t]$. Then, for every $t^*\in(a,b]$, we have*

$$\int_a^{t^*}\Im\left[\frac{A'_{(0)}(t)+jA'_{(1)}(t)}{A_{(0)}(t)+jA_{(1)}(t)}\right]dt$$

$$=\left\{\begin{array}{ll}\arctan\left\{\frac{A_{(1)}(t^*)}{A_{(0)}(t^*)}\right\}+[V\{\Psi(t^*)\}-V\{\Psi(a)\}]\pi & \\ & \textit{if } A_{(0)}(t^*)\neq 0,\\ \pi/2+[V\{\Psi(t^*)\}-V\{\Psi(a)\}]\pi & \textit{if } A_{(0)}(t^*)=0,\end{array}\right\}$$

$$-\begin{cases}\arctan\left\{\frac{A_{(1)}(a)}{A_{(0)}(a)}\right\} & \textit{if } A_{(0)}(a)\neq 0,\\ \operatorname{sgn}(\Psi_0(a)\Psi_1(a))\pi/2 & \textit{if } A_{(0)}(a)=0,\end{cases}$$

where $\operatorname{sgn}(t)=t/|t|$ for $t\neq 0$, $\operatorname{sgn}(t)=0$ for $t=0$, and $V\{\Psi(t)\}$ is the number of sign changes in the polynomials $\{\Psi_0(t),\Psi_1(t),\ldots,\Psi_q(t)\}$ generated by Algorithm 1.

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