2次元スプライン平滑化に基づく代数的連続位相復元法

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Algebraic Phase Unwrapping Based on Two-Dimensional Spline Smoothing

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Abstract– In this report, we introduce the main ideas of our approach [Kitahara and Yamada, IEEE Transactions on Signal Processing, (accepted for publication)] for high-resolution 2D phase unwrapping. In the first step (SPS: SPline Smoothing), we construct a pair of the smoothest spline functions which minimize the energies of their local changes while interpolating respectively the cosine and the sine of given wrapped phase. If these functions have no common zero over the domain of our interest, the proposed estimate of the unwrapped phase can be obtained by algebraic phase unwrapping in the second step (APU: Algebraic Phase Unwrapping) as a continuous function. The smoothness of the proposed unwrapped phase function is guaranteed globally over the domain without losing any consistency with the wrapped phase. Numerical experiments for terrain height estimation demonstrate the effectiveness of the proposed method.

Key Words: Two-dimensional phase unwrapping, Algebraic phase unwrapping, Bivariate Spline smoothing, Convex optimization, Interferometric synthetic aperture radar, Terrain height estimation.

1 Introduction

Two-dimensional (2D) phase unwrapping[1), 2)] is an estimation problem of an unknown continuous phase function $\Theta : \Omega \to \mathbb{R}$ from its noisy wrapped samples

$$\Theta^W(x, y) := W(\Theta(x, y) + \nu(x, y)) \in (-\pi, \pi] \qquad (1)$$

observed at $(x, y) \in \mathcal{G}$ $(\subset \Omega)$, where Ω $(\subset \mathbb{R}^2)$ is a simply connected closed region, $\mathcal{G}$ is the set of finite sampling points, ν is additive phase noise, and $W : \mathbb{R} \to (-\pi, \pi]$ is the *wrapping operator* defined by

$$\forall \vartheta \in \mathbb{R}\ \exists \eta \in \mathbb{Z} \quad \vartheta = W(\vartheta) + 2\pi\eta \text{ and } W(\vartheta) \in (-\pi, \pi].$$

Θ and Θ^W are respectively called the *unwrapped phase* and the *wrapped phase*. 2D phase unwrapping is important for signal and image processing applications such as *terrain height estimation* and *landslide identification* by interferometric synthetic aperture radar (InSAR)[3)–10)], *seafloor depth estimation* by interferometric synthetic aperture sonar (InSAS)[11)–14)], *3D shape measurement* by fringe projection[15)–18)] or X-ray[19)–22)], and *water/fat separation* in magnetic resonance imaging (MRI)[23)–26)].

As remarked clearly in [27], all commonly used phase unwrapping algorithms are based on the assumption that the true unwrapped phase field varies slowly enough that in most places, neighboring phase values are within one-half cycle (π rad) of one another, i.e., it is assumed that $\Delta\Theta_{\boldsymbol{i}} := \Theta(\widetilde{x}, \widetilde{y}) - \Theta(x, y) \in (-\pi, \pi]$ for most neighboring pairs of samples $\boldsymbol{i} := ((x, y), (\widetilde{x}, \widetilde{y})) \in \mathcal{G} \times \mathcal{G}$. Such algorithms have been designed to suppress a certain function J measuring the unwrapped phase differences $\Delta\Theta_{\boldsymbol{i}}$ for all neighboring pairs $\boldsymbol{i} \in \mathcal{G} \times \mathcal{G}$ as

$$J(\boldsymbol{\Theta}) := \sum_{\boldsymbol{i}} w_{\boldsymbol{i}} \left| \Delta\Theta_{\boldsymbol{i}} - W\left(\Delta\Theta_{\boldsymbol{i}}^W\right) \right|^p,$$

where $w_{\boldsymbol{i}} > 0$, $p > 0$, $\boldsymbol{\Theta} := \mathrm{vec}(\Theta(x, y))_{(x,y)\in\mathcal{G}}$ stands for the vectorization of $\Theta(x, y)$ on $\mathcal{G}$, and $\Delta\Theta_{\boldsymbol{i}}^W := \Theta^W(\widetilde{x}, \widetilde{y}) - \Theta^W(x, y)$ is the wrapped phase difference between a neighboring pair of samples $\boldsymbol{i} = ((x, y), (\widetilde{x}, \widetilde{y}))$. For example, branch cut (BC) algorithm[5)] and minimum spanning tree (MST) algorithm[27)] employ $p \to +0$, minimum cost flow (MCF) algorithm[28)] employs $p = 1$, and least squares (LS) algorithm[29)] employs $p = 2$. Such a specification of J is introduced on the basis of a simple property that, under the assumption $\nu = 0$,

$$\Delta\Theta_{\boldsymbol{i}} = W(\Delta\Theta_{\boldsymbol{i}}^W) \Leftrightarrow \Delta\Theta_{\boldsymbol{i}} \in (-\pi, \pi].$$

Then the algorithms try to use a minimizer of J as an estimate of the unwrapped phase.

BC, MST and MCF algorithms assume that noise ν in (1) is small enough and try to find a minimizer of J under the condition

$$\forall (x, y) \in \mathcal{G} \quad W(\Theta(x, y)) = \Theta^W(x, y) \qquad (2)$$

This type of optimization problem is combinatorial and intractable due to condition (2). In order to solve this problem, these algorithms use an elegant technique developed originally for network flow in graph theory[2), 27)]. In this approach, if the observed wrapped phase has only small noise and the unwrapped phase difference is sufficiently small with respect to sampling interval, we can construct a very good estimate. However, otherwise, condition (2) is violated due to noise ν in (1), and the minimizer of J is hard to compute due to condition (2).

LS algorithm directly computes a minimizer $\boldsymbol{\Theta}^*$ of J without requiring condition (2). In this approach, even if the observed wrapped phase is noisy, $\boldsymbol{\Theta}^*$ can be obtained. However the consistency between Θ^* and Θ^W, i.e., $W(\Theta^*(x, y)) \approx \Theta^W(x, y)$ is not guaranteed at many sampling points $(x, y) \in \mathcal{G}$.

In this paper, we propose a completely different algebraic approach to 2D phase unwrapping by exploiting the property of $\Theta^W \in (-\pi, \pi]$:

$$\Theta^W = W(\Theta + \nu)$$
$$\Leftrightarrow \left(\cos\Theta^W, \sin\Theta^W\right) = \left(\cos(\Theta + \nu), \sin(\Theta + \nu)\right). \qquad (3)$$

The proposed scheme achieves a high-resolution estimate of the unwrapped phase Θ unlike many existing algorithms[5), 10), 25)–29)]. We estimate Θ as the continuous phase function $\theta_f \in C^2(\Omega)$ of a twice continuously differentiable complex function $f := f_{(0)} + \imath f_{(1)} = |f|e^{\imath\theta_f}$, where $f_{(0)} \in C^2(\Omega)$ and $f_{(1)} \in C^2(\Omega)$ have no common zero over Ω (see Notation in the end of this section). Then the estimation problem of Θ is replaced with those of $f_{(0)}$ and $f_{(1)}$ which respectively approximate $\cos\Theta$ and $\sin\Theta$. Clearly, by (3), $f_{(0)}$ and $f_{(1)}$ are desired to interpolate respectively $\cos(\Theta^W(x,y)) = \cos(\Theta(x,y))$ and $\sin(\Theta^W(x,y)) = \sin(\Theta(x,y))$ if $\nu(x,y) = 0$ at $(x,y) \in \mathcal{G}$. Motivated by the main idea of *functional data analysis*[30)–32)], we assume that f is "smooth" which means that the energy of local change is small over Ω, and adopt the *bivariate spline space* as the set of all candidates of $f_{(0)}$ and $f_{(1)}$. After finding the smoothest spline functions $f^*_{(0)}$ and $f^*_{(1)}$ which are consistent with the wrapped phase information $\cos(\Theta^W(x,y))$ and $\sin(\Theta^W(x,y))$ at $(x,y) \in \mathcal{G}$ (*SPline Smoothing (SPS)*), the continuous phase function θ_{f^*} of $f^* := f^*_{(0)} + \imath f^*_{(1)} = |f^*|e^{\imath\theta_{f^*}}$ is analytically computed, as the proposed estimate of Θ, by *Algebraic Phase Unwrapping (APU)*[33)–37)]. This approach is particularly effective in the case where phase noise ν is relatively small and f^* has no zero over Ω. Indeed, by this approach, we can maximize a certain smoothness of θ_f subject to the condition $W(\theta_f(x,y)) \approx \Theta^W(x,y)$ for all sampling points $(x,y) \in \mathcal{G}$ unlike other algorithms. Numerical experiments for InSAR terrain height estimation demonstrate the effectiveness of the proposed scheme.

Notation Let $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ be the set of all integers, non-negative integers, real numbers, non-negative real numbers, and complex numbers, respectively. We use $\imath \in \mathbb{C}$ to denote the imaginary unit, i.e., $\imath^2 = -1$, and use $i, j \in \mathbb{Z}_+$ for general indices. For any set S, $\mathrm{card}(S)$ stands for its cardinal number. For $\rho \in \mathbb{Z}_+$, $C^\rho(\Omega)$ stands for the set of all ρ-times continuously differentiable real-valued functions over a simply connected region $\Omega \subset \mathbb{R}^2$. A boldface letter expresses a vector or a matrix.

2 Preliminaries—Bivariate Spline Function

We restrict ourselves to partitioning a polygonal domain $\Omega \subset \mathbb{R}^2$ into triangles because these have the most flexibility with respect to the resolution of discretization in Ω.

Define a triangle $\mathcal{T} \subset \mathbb{R}^2$, by specifying three vertices $\boldsymbol{v}_k := (x_k, y_k) \in \mathbb{R}^2$ $(k = 1,2,3)$ which are not arranged linearly, i.e., $\varrho := x_1y_2 - y_1x_2 + x_2y_3 - y_2x_3 + x_3y_1 - y_3x_1 \neq 0$, as

$$\mathcal{T} := \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle := \left\{ r\boldsymbol{v}_1 + s\boldsymbol{v}_2 + t\boldsymbol{v}_3 \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} r,s,t \in [0,1] \\ r+s+t = 1 \end{array} \right\}.$$

Let $\Delta := \{\mathcal{T}_i\}_{i=1}^N$ be a collection of triangles $\mathcal{T}_i \subset \mathbb{R}^2$ whose union forms a simply connected closed region $\Omega \subset \mathbb{R}^2$, i.e., $\bigcup_{i=1}^N \mathcal{T}_i = \Omega$. If, for any pair of triangles $\mathcal{T}_i \in \Delta$ and $\mathcal{T}_j \in \Delta$ $(i \neq j)$, $\mathcal{T}_i \cap \mathcal{T}_j$ is either empty or a common edge or a common vertex, the collection Δ is called a *regular triangulation*. Given a regular triangulation Δ and $\rho, d \in \mathbb{Z}_+$ s.t. $0 \le \rho < d$, define

$$\mathcal{S}^\rho_d(\Delta) := \{ f \in C^\rho(\Omega) \mid \forall \mathcal{T}_i \in \Delta \quad f = f_i \in \mathbb{P}_d \text{ over } \mathcal{T}_i \}$$

as the set of all bivariate spline functions of degree d and smoothness ρ on Δ, where $\mathbb{P}_d$ stands for the set of all bivariate polynomials whose degree is d at most, i.e., $\mathbb{P}_d := \{ f : \mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto \sum_{i=0}^d \sum_{j=0}^{d-i} c_{i,j} x^i y^j \mid c_{i,j} \in \mathbb{R} \}$.

For $\mathcal{T} = \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle$ s.t. $\boldsymbol{v}_k = (x_k, y_k) \in \mathbb{R}^2$ $(k = 1,2,3)$, every $(x,y) \in \mathbb{R}^2$ can be expressed in the form

$$(x,y) = r\boldsymbol{v}_1 + s\boldsymbol{v}_2 + t\boldsymbol{v}_3 \quad \text{s.t. } r+s+t = 1,$$

where (r,s,t) is called *barycentric coordinate*[38), 39)] of (x,y) with respect to $\mathcal{T}$ and expressed as

$$\left.\begin{array}{l} r = ((y_2 - y_3)x - (x_2 - x_3)y + x_2y_3 - y_2x_3)/\varrho \\ s = ((y_3 - y_1)x - (x_3 - x_1)y + x_3y_1 - y_3x_1)/\varrho \\ t = ((y_1 - y_2)x - (x_1 - x_2)y + x_1y_2 - y_1x_2)/\varrho \end{array}\right\}.$$

By using the above expression of (x,y), the *Bernstein-Bézier polynomial* of degree d is defined, for $\mathcal{T}$ and $(l,m,n) \in \mathbb{Z}^3_+$ satisfying $l+m+n = d$, as

$$B^{\mathcal{T}}_{l,m,n} : \mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto \frac{d!}{l!m!n!} r^l s^m t^n.$$

It is known that $\{B^{\mathcal{T}}_{l,m,n} \mid l,m,n \in \mathbb{Z}_+ \text{ and } l+m+n = d\}$ is a basis of $\mathbb{P}_d$, and hence any piecewise polynomial f, whose restriction f_i to $\mathcal{T}_i \in \Delta$ satisfies $f_i \in \mathbb{P}_d$ $(i = 1,2,\ldots,N)$, can be expressed uniquely as

$$f_i(x,y) = \sum_{l+m+n=d} c^{\mathcal{T}_i}_{l,m,n} \frac{d!}{l!m!n!} r^l s^m t^n,$$

where (r,s,t) is barycentric coordinate with respect to $\mathcal{T}_i$. Such a representation of piecewise polynomials is called the *Bernstein-Bézier form*, and $c^{\mathcal{T}_i}_{l,m,n} \in \mathbb{R}$ is called the *Bernstein-Bézier coefficient* (or *B-coefficient*). We define the B-coefficient vector as $\boldsymbol{c} := \mathrm{vec}(c^{\mathcal{T}_i}_{l,m,n})^{i=1,2,\ldots,N}_{l+m+n=d}$.

3 Algebraic Recovery of Unwrapped Phase

3.1 General Idea of The Proposed Scheme

In this section, we propose an algebraic approach for high-resolution 2D phase unwrapping. We estimate the unwrapped phase Θ as a continuous function defined over Ω unlike many existing algorithms. In our previous works[36), 37)], by using *Poincaré's lemma*[40)], we clarified the condition for the unique existence of the continuous phase function $\theta_f \in C^2(\Omega)$ of a complex function $f := f_{(0)} + \imath f_{(1)} : \Omega \to \mathbb{C}$ s.t. $f_{(k)} \in C^2(\Omega)$ $(k = 0,1)$.

Fact 1 ([36]) *Let Ω be a simply connected closed region on $\mathbb{R}^2$. Suppose that $f_{(k)} : \Omega \to \mathbb{R}$ $(k = 0,1)$ are twice continuously differentiable functions, i.e., $f_{(k)} \in C^2(\Omega)$, and satisfy $f(x,y) := f_{(0)}(x,y) + \imath f_{(1)}(x,y) \neq 0$ for all $(x,y) \in \Omega$. Then for arbitrarily fixed $(x_0,y_0) \in \Omega$ and θ_0 satisfying $f(x_0,y_0) = |f(x_0,y_0)|e^{\imath\theta_0}$, the following hold.*

(i) There exists a unique continuous function $\theta_f \in C^2(\Omega)$ satisfying $\theta_f(x_0, y_0) = \theta_0$ and

$$\left.\begin{aligned} \frac{\partial \theta_f}{\partial x}(x,y) &= \Im\left[\frac{\frac{\partial f_{(0)}}{\partial x}(x,y) + \imath \frac{\partial f_{(1)}}{\partial x}(x,y)}{f_{(0)}(x,y) + \imath f_{(1)}(x,y)}\right] \\ \frac{\partial \theta_f}{\partial y}(x,y) &= \Im\left[\frac{\frac{\partial f_{(0)}}{\partial y}(x,y) + \imath \frac{\partial f_{(1)}}{\partial y}(x,y)}{f_{(0)}(x,y) + \imath f_{(1)}(x,y)}\right] \end{aligned}\right\} \quad (4)$$

for all $(x, y) \in \Omega$, where $\Im(c)$ stands for the imaginary part of $c \in \mathbb{C}$. θ_f satisfies

$$f(x,y) = |f(x,y)|e^{\imath\theta_f(x,y)} \quad \textit{for all } (x,y) \in \Omega.$$

(ii) Let $\Upsilon : [a, b] \to \Omega$ be a piecewise C^1 path s.t. $\Upsilon(a) = (x_0, y_0)$ and $\Upsilon(b) = (x_1, y_1) \in \Omega$. Then we have

$$\theta_f(x_1, y_1) = \theta_0 + \int_a^b \Im\left[\frac{F'_{(0)}(\tau) + \imath F'_{(1)}(\tau)}{F_{(0)}(\tau) + \imath F_{(1)}(\tau)}\right] \mathrm{d}\tau,$$

where $F_{(k)}(\tau) := f_{(k)}(\Upsilon(\tau))$ $(k = 0, 1)$. □

Remark 1 (Note on Equation (4)) *Note that*

$$\left.\begin{aligned} \Im\left[\frac{\frac{\partial f_{(0)}}{\partial x}(x,y)+\imath\frac{\partial f_{(1)}}{\partial x}(x,y)}{f_{(0)}(x,y)+\imath f_{(1)}(x,y)}\right] &= \frac{\partial}{\partial x}\left[\arctan\left(\frac{f_{(1)}(x,y)}{f_{(0)}(x,y)}\right)\right] \\ \Im\left[\frac{\frac{\partial f_{(0)}}{\partial y}(x,y)+\imath\frac{\partial f_{(1)}}{\partial y}(x,y)}{f_{(0)}(x,y)+\imath f_{(1)}(x,y)}\right] &= \frac{\partial}{\partial y}\left[\arctan\left(\frac{f_{(1)}(x,y)}{f_{(0)}(x,y)}\right)\right] \end{aligned}\right\}$$

holds at every $(x, y) \in \Omega$ satisfying $f_{(0)}(x, y) \neq 0$, where $\arctan(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ denotes the principle value of the inverse tangent for all $x \in \mathbb{R}$, i.e., $\tan(\arctan(x)) = x$. □

Trying to estimate Θ by $\theta_f \in C^2(\Omega)$, from Fact 1, we can reduce the estimation problem of Θ to those of $f_{(0)} \in C^2(\Omega)$ and $f_{(1)} \in C^2(\Omega)$ which respectively approximate $\cos\Theta$ and $\sin\Theta$. In particular, under the assumption that phase noise ν is not significant in (1), $f_{(0)}$ and $f_{(1)}$ are desired to interpolate $\cos(\Theta^W(x, y)) \approx \cos(\Theta(x, y))$ and $\sin(\Theta^W(x, y)) \approx \sin(\Theta(x, y))$, respectively, at every sampling point $(x, y) \in \mathcal{G}$. Moreover, on the basis of the idea of *functional data analysis*[30)–32)], we search for $f_{(0)}$ and $f_{(1)}$ which are smooth. Here the word "smooth" means that the energy of local change, i.e., the ℓ_2 norm of the second order partial derivative, is small over Ω. Therefore we design a smooth continuous phase function θ_f, by minimizing the energy of local change of $f_{(k)}$ $(k = 0, 1)$:

$$\iint_\Omega \left[\left|\frac{\partial^2 f_{(k)}}{\partial x^2}\right|^2 + 2\left|\frac{\partial^2 f_{(k)}}{\partial x \partial y}\right|^2 + \left|\frac{\partial^2 f_{(k)}}{\partial y^2}\right|^2\right] \mathrm{d}x\mathrm{d}y \quad (5)$$

in a suitable functional space subject to $|f(x, y)| > 0$ for all $(x, y) \in \Omega$ and[1]

$$\left.\begin{aligned} f_{(0)}(x,y) &= \cos\big(\Theta^W(x,y)\big) \\ f_{(1)}(x,y) &= \sin\big(\Theta^W(x,y)\big) \end{aligned}\right\} \quad \text{for all } (x,y) \in \mathcal{G}. \quad (6)$$

[1]Of course, condition (6) can be generalized in a natural way if amplitude information at every sampling point $(x, y) \in \mathcal{G}$ is available.

We can guarantee $W(\theta_f(x, y)) = \Theta^W(x, y)$ for all sampling points $(x, y) \in \mathcal{G}$ if (6) and $|f(x, y)| > 0$ for all $(x, y) \in \Omega$. Motivated by Fact 1 and the successful utilization of spline functions in functional data analysis[41)–46)], we adopt the bivariate spline space $\mathcal{S}_d^2(\Delta)$ $(d \geq 3)$ as the set of all possible candidates of $f_{(k)}$.

As a result, we propose the following 2D phase unwrapping scheme whose core consists of *SPline Smoothing (SPS)* and *Algebraic Phase Unwrapping (APU)*.

SPS: Find $f^*_{(k)} \in \mathcal{S}_d^2(\Delta) \subset C^2(\Omega)$ $(k = 0, 1$ and $d \geq 3)$ which minimize (5) subject to (6).

APU: For any point of interest $(x, y) \in \Omega$, compute the value of $\theta_{f^*}(x, y)$ defined in Fact 1(ii) along a suitable piecewise C^1 path Υ.

Note that SPS is a convex relaxation of an original optimization problem, defined with (5) and (6), which requires an additional condition $f_{(0)}(x, y) + \imath f_{(1)}(x, y) \neq 0$ for all $(x, y) \in \Omega$. Fortunately, if the observed wrapped phase Θ^W is not contaminated by severe phase noise and sufficiently many sampling points are available to capture the geometric feature of Θ, the solution $(f^*_{(0)}, f^*_{(1)})$ of this relaxed problem tends to automatically satisfy the additional condition. If there exists some $(x, y) \in \Omega$ s.t. $f^*_{(0)}(x, y) + \imath f^*_{(1)}(x, y) = 0$, we use a denoising step proposed in Section 3.4 to avoid the occurrence of zeros.

3.2 SPline Smoothing (SPS)

Let $\boldsymbol{c}_{(k)}$ $(k = 0, 1)$ be the B-coefficient vectors of $f_{(k)} \in \mathcal{S}_d^2(\Delta)$ (see Section 2). Then the energy of local change in (5) can be expressed as $\boldsymbol{c}_{(k)}^T \boldsymbol{Q} \boldsymbol{c}_{(k)}$, where $\boldsymbol{Q}$ is a symmetric positive semidefinite matrix[47)]. The condition $f_{(k)} \in \mathcal{S}_d^2(\Delta)$ is equivalent to $\boldsymbol{H}\boldsymbol{c}_{(k)} = \boldsymbol{0}$ and condition (6) can be expressed as $\boldsymbol{\mathcal{I}}\boldsymbol{c}_{(k)} = \boldsymbol{d}_{(k)}$ in terms of

$$\left.\begin{aligned} \boldsymbol{d}_{(0)} &:= \mathrm{vec}\big(\cos\big(\Theta^W(x,y)\big)\big)_{(x,y)\in\mathcal{G}} \\ \boldsymbol{d}_{(1)} &:= \mathrm{vec}\big(\sin\big(\Theta^W(x,y)\big)\big)_{(x,y)\in\mathcal{G}} \end{aligned}\right\}$$

and a sparse matrix $\boldsymbol{\mathcal{I}}$[46)]. Indeed, if we assume that

$$\text{every } (x,y) \in \mathcal{G} \text{ is a vertex of some } \mathcal{T} \in \Delta, \quad (7)$$

each row vector of $\boldsymbol{\mathcal{I}}$ has only one non-zero component '1'. As a result, SPS in the proposed scheme is reduced to the following convex optimization problem, say SPS again, for the B-coefficient vector $\boldsymbol{c}_{(k)}$:

SPS: Find $\boldsymbol{c}^*_{(k)}$ $(k = 0, 1)$ minimizing

$$\boldsymbol{c}_{(k)}^T \boldsymbol{Q} \boldsymbol{c}_{(k)}$$
$$\text{subject to } \boldsymbol{H}\boldsymbol{c}_{(k)} = \boldsymbol{0} \text{ and } \boldsymbol{\mathcal{I}}\boldsymbol{c}_{(k)} = \boldsymbol{d}_{(k)}.$$

Moreover, by considering the influence of phase noise ν, we can relax SPS as a generalized Hermite-Birkhoff interpolation problem[44)]:

SPS+: Find $\boldsymbol{c}^*_{(k)}$ $(k = 0, 1)$ minimizing

$$\boldsymbol{c}_{(k)}^T \boldsymbol{Q} \boldsymbol{c}_{(k)}$$
$$\text{subject to } \boldsymbol{H}\boldsymbol{c}_{(k)} = \boldsymbol{0} \text{ and } -\boldsymbol{\epsilon}_{(k)} \leq \boldsymbol{\mathcal{I}}\boldsymbol{c}_{(k)} - \boldsymbol{d}_{(k)} \leq \boldsymbol{\epsilon}_{(k)},$$

where $\boldsymbol{\epsilon}_{(k)} := \mathrm{vec}(\epsilon_{(k)}(x,y))_{(x,y)\in\mathcal{G}} \in \mathbb{R}_{+}^{\mathrm{card}(\mathcal{G})}$ $(k = 0,1)$ are the acceptable interpolation errors designed to be small if the wrapped phase $\Theta^W(x,y)$ is reliable at $(x,y) \in \mathcal{G}$, and relatively large otherwise. SPS and SPS+ can be solved by quadratic programming solvers[48)–50)], if the constraints are feasible.

Even if the constraint in SPS (or SPS+) is infeasible, it can be relaxed in the following sense of hierarchical convex optimization problem:

SPS++: Find $\boldsymbol{c}_{(k)}^{**}$ $(k = 0,1)$ minimizing

$$\boldsymbol{c}_{(k)}^{*T}\boldsymbol{Q}\boldsymbol{c}_{(k)}^{*}$$

$$\text{subject to } \boldsymbol{c}_{(k)}^{*} \in \underset{\boldsymbol{H}\boldsymbol{c}_{(k)}=\boldsymbol{0}}{\mathrm{argmin}} \|\boldsymbol{\mathcal{I}}\boldsymbol{c}_{(k)} - \boldsymbol{d}_{(k)}\|_2^2.$$

SPS++ is solved by *hybrid steepest descent method*[51)–56)].

3.3 Algebraic Phase Unwrapping (APU)

Let $\Delta := \{\mathcal{T}_i := \langle \boldsymbol{v}_1^{\langle i\rangle}, \boldsymbol{v}_2^{\langle i\rangle}, \boldsymbol{v}_3^{\langle i\rangle}\rangle\}_{i=1}^N$ be a regular triangulation satisfying (7), and let $\theta_0 \in \mathbb{R}$ satisfy $f^*(\boldsymbol{v}_1^{\langle 1\rangle}) := f_{(0)}^*(\boldsymbol{v}_1^{\langle 1\rangle}) + \imath f_{(1)}^*(\boldsymbol{v}_1^{\langle 1\rangle}) = |f^*|e^{\imath\theta_0}$. Suppose that we are interested in θ_{f^*} of f^* at $\boldsymbol{v}_2^{\langle K\rangle}$ $(1 \le K \le N)$, where we assume, without loss of generality, $\boldsymbol{v}_1^{\langle i+1\rangle} = \boldsymbol{v}_2^{\langle i\rangle}$ $(i = 1,2,\ldots,K-1)$ by renumbering the indices of triangles and their vertices if necessary. Define a piecewise C^1 path $\Upsilon : [0,K] \to \bigcup_{i=1}^K \mathcal{T}_i$ by

$$\Upsilon(\tau) := (\tau - i + 1)(\boldsymbol{v}_2^{\langle i\rangle} - \boldsymbol{v}_1^{\langle i\rangle}) + \boldsymbol{v}_1^{\langle i\rangle} \quad \text{for } \tau \in [i-1, i],$$

and then, from Fact 1(ii), $\theta_{f^*}(\boldsymbol{v}_2^{\langle K\rangle})$ is expressed as

$$\begin{aligned}\theta_{f^*}(\boldsymbol{v}_2^{\langle K\rangle}) &= \theta_0 + \int_0^K \Im\left[\frac{F'_{(0)}(\tau) + \imath F'_{(1)}(\tau)}{F_{(0)}(\tau) + \imath F_{(1)}(\tau)}\right] \mathrm{d}\tau, \\ &= \theta_0 + \sum_{i=1}^K \int_0^1 \Im\left[\frac{F_{(0)}^{\langle i\rangle\prime}(\tau) + \imath F_{(1)}^{\langle i\rangle\prime}(\tau)}{F_{(0)}^{\langle i\rangle}(\tau) + \imath F_{(1)}^{\langle i\rangle}(\tau)}\right] \mathrm{d}\tau, \end{aligned} \quad (8)$$

where $F_{(k)}(\tau) := f_{(k)}^*(\Upsilon(\tau))$ $(k = 0,1)$ and $F_{(k)}^{\langle i\rangle}(\tau) := F_{(k)}(\tau + i - 1) = f_{(k)}^*(\Upsilon(\tau + i - 1))$ $(\tau \in [0,1], k = 0,1$ and $i = 1,2,\ldots,K)$. Since $F_{(k)}^{\langle i\rangle}(\tau) \in \mathbb{R}[\tau]$ $(k = 0,1)$ are univariate polynomials of degree d at most, all integrals in (8) can be computed analytically by the following method called *algebraic phase unwrapping*[33)–37)].

Fact 2 ([36]) *Let $P_{(k)}(\tau)$ $(k = 0,1)$ be univariate real polynomials, and let $P(\tau) := P_{(0)}(\tau) + \imath P_{(1)}(\tau)$ be a univariate complex polynomial satisfying $P(\tau) \neq 0$ for all $\tau \in [a,b]$. Then, for every $\tau^* \in (a,b]$, we have*

$$\begin{aligned}&\int_a^{\tau^*} \Im\left[\frac{P'_{(0)}(\tau) + \imath P'_{(1)}(\tau)}{P_{(0)}(\tau) + \imath P_{(1)}(\tau)}\right] \mathrm{d}\tau \\ &= \begin{cases} \arctan(\mathcal{Q}(\tau^*)) + \left[V(\Psi(\tau^*)) - V(\Psi(a))\right]\pi & \\ \qquad\qquad \text{if } P_{(0)}(\tau^*) \neq 0; \\ \dfrac{\pi}{2} + \left[V(\Psi(\tau^*)) - V(\Psi(a))\right]\pi & \text{if } P_{(0)}(\tau^*) = 0; \end{cases} \\ &\quad - \begin{cases} \arctan(\mathcal{Q}(a)) & \text{if } P_{(0)}(a) \neq 0; \\ \mathrm{sgn}(\Psi_0(a)\Psi_1(a))\dfrac{\pi}{2} & \text{if } P_{(0)}(a) = 0, \end{cases} \end{aligned} \quad (9)$$

Input: $P_{(0)}(\tau) \in \mathbb{R}[\tau]$, $P_{(1)}(\tau) \in \mathbb{R}[\tau]$ and $a \in \mathbb{R}$
Output: $(\Psi_j(\tau))_{j=0}^q$

```
Ψ_0(τ) ← P_(0)(τ)/(τ−a)^{e_0}   (e_0: order of a as a zero of polynomial P_(0))
Ψ_1(τ) ← P_(1)(τ)/(τ−a)^{e_1}   (e_1: order of a as a zero of polynomial P_(1))
j ← 1
while deg(Ψ_j) ≥ 1   (deg(Ψ_j): degree of polynomial Ψ_j) do
    Ψ_{j+1} ← −rem(Ψ_{j−1}, Ψ_j)
    (rem(Ψ_{j−1}, Ψ_j): remainder of division of Ψ_{j−1} by Ψ_j)
    j ← j + 1
end while
q ← j
Return (Ψ_j(τ))_{j=0}^q
```

Fig. 1: Algorithm generating $(\Psi_j(\tau))_{j=0}^q$ in Fact 2.

where $\mathcal{Q}(\tau) := P_{(1)}(\tau)/P_{(0)}(\tau)$, $\mathrm{sgn}(x) := x/|x|$ for $x \neq 0$, $\mathrm{sgn}(x) := 0$ for $x = 0$, and $V(\Psi(\tau^*)), V(\Psi(a)) \in \mathbb{Z}_+$ are the numbers of sign changes, at $\tau = \tau^*$ and $\tau = a$, in the polynomial sequence $(\Psi_j(\tau))_{j=0}^q$ generated by the algorithm in Fig. 1 (e.g., if $q = 5$, $\tau^* = 1$ and $(\Psi_0(1), \Psi_1(1), \Psi_2(1), \Psi_3(1), \Psi_4(1), \Psi_5(1)) = (3,-2,5,1,0,-2)$, $V(\Psi(\tau^*)) = 3$ because there are three sign changes $(3 \to -2)$, $(-2 \to 5)$ and $(1 \to -2)$). □

In [36], we also proposed an alternative way, based on *subresultant* theory[57)], of computation for $V(\Psi(\tau^*))$ and $V(\Psi(a))$ in (9), to resolve certain numerical instabilities caused by polynomial division in the algorithm in Fig. 1. In this report, we use [36, Theorem3] for fast and stable evaluations of $V(\Psi(\tau^*))$ and $V(\Psi(a))$ in (9).

Note that, under the condition $f^*(x,y) \neq 0$ for all $(x,y) \in \Omega$, we can compute $\theta_{f^*}(x,y)$ not only at $(x,y) \in \mathcal{G}$ but also at any $(x,y) \in \Omega$ by repeatedly applying algebraic phase unwrapping. Therefore, unlike many existing algorithms, the proposed scheme gives a smooth θ_{f^*}, as a high-resolution estimate of Θ, which is consistent with the wrapped phase, i.e., $W(\theta_{f^*}(x,y)) \approx \Theta^W(x,y)$ at $(x,y) \in \mathcal{G}$. This approach is particularly effective in the case where phase noise is relatively small.

3.4 Denoising by Selective Smoothing (DSS)

It is well-known that phase noise observed at even small portion of sampling points can create residues which influence the global feature of the results of existing 2D phase unwrapping algorithms[1), 2), 58)–60)]. This has been a central reason of the difficulty in 2D phase unwrapping. In the proposed scheme for noisy wrapped samples, the occurrence of common zeros of $f_{(0)}^*$ and $f_{(1)}^*$ in SPS (or SPS+ or SPS++), which yields the path dependency of θ_{f^*} in APU, can be seen as such a type of difficulty. These facts suggest that excessive fidelity to noisy wrapped samples easily leads to poor estimates in 2D phase unwrapping problem.

To suppress the influence of noise, we denoise the wrapped phase $\Theta^W(x,y)$ to obtain $\widetilde{\Theta}^W(x',y') \in (-\pi,\pi]$ $((x',y') \in \mathcal{G}' \supset \mathcal{G})$ by smoothing Θ^W while keeping the condition $\widetilde{\Theta}^W(x,y) = \Theta^W(x,y)$ for all $(x,y) \in \mathcal{G}_{\mathrm{I}}$ $(\subset \mathcal{G})$, where $\mathcal{G}_{\mathrm{I}}$ is the set of all reliable sampling points. The reliability of each sampling point is judged on the basis of the wrapped phase difference and the residues. The smoothing is realized by using convex optimization. The main idea of *Denoising by Selective Smoothing (DSS)* is divided into the following two substeps.

DSS-1: Classify all sampling points in $\mathcal{G}$ into $\mathcal{G}_{\mathrm{I}}$ (Type I: reliable) and $\mathcal{G}_{\mathrm{II}} := \mathcal{G} \setminus \mathcal{G}_{\mathrm{I}}$ (Type II: unreliable) by using the information of $W(\Delta\Theta_{\boldsymbol{i}}^W)$ and residues.

DSS-2: Produce smoothed wrapped samples $\widetilde{\Theta}^W(x,y) \in (-\pi,\pi]$ at $(x,y) \in \mathcal{G}'$ $(\supset \mathcal{G})$, where $\widetilde{\Theta}^W$ satisfies

$$\widetilde{\Theta}^W(x,y) = \Theta^W(x,y) \quad \text{if } (x,y) \in \mathcal{G}_{\mathrm{I}}$$

and $\widetilde{\Theta}^W(x,y)$ at $(x,y) \in \mathcal{G}' \setminus \mathcal{G}_{\mathrm{I}}$ is determined by interpolation of a minimizer of the following convex function:

$$\widetilde{J}(\boldsymbol{\Theta}) := \|\boldsymbol{D}_1\boldsymbol{\Theta} - \boldsymbol{\delta}\|_{1,\boldsymbol{w}_1} + \|\boldsymbol{D}_2\boldsymbol{\Theta}\|_{2,\boldsymbol{w}_2}^2,$$

where we express $\sum_{\boldsymbol{i}} w_{\boldsymbol{i}} |\Delta\Theta_{\boldsymbol{i}} - W(\Delta\Theta_{\boldsymbol{i}}^W)|$ in J as $\|\boldsymbol{D}_1\boldsymbol{\Theta} - \boldsymbol{\delta}\|_{1,\boldsymbol{w}_1}$ in $\widetilde{J}$, and $\|\boldsymbol{D}_2\boldsymbol{\Theta}\|_{2,\boldsymbol{w}_2}^2$ stands for the square of an weighted ℓ_2 norm of the second order differences of $\boldsymbol{\Theta}$.

For more details on DSS, see [61].

4 Application to Terrain Height Estimation

In this section, we apply the proposed 2D phase unwrapping scheme to terrain height estimation by InSAR.

4.1 Terrain Height Estimation by InSAR

Interferometric synthetic aperture radar (InSAR)[3)–9)] is an imaging technique allowing highly accurate measurements of surface topography in all weather conditions, day or night. In InSAR system (see Fig. 2(a)), Antenna 1 and Antenna 2 on-board an aircraft or a spacecraft platform transmit coherent broadband radio signals and receive the reflected signals $s_k := |s_k| e^{-\imath(\frac{4\pi R_k}{\lambda} + \phi_k + \nu_k)}$ $(k = 1,2)$ from a target corresponding to $(x,y) \in \Omega \subset \mathbb{R}^2$, where λ is the wavelength of the transmitted signal, R_k is the distance from Antenna k to the target, ϕ_k is the backscatter phase delay, ν_k is additive phase noise, and the dependencies of variables $R_k, \phi_k, \nu_k, \theta_o$ and θ_i on (x,y) are omitted for notational simplicity in Fig. 2 and in the discussion below. Since the backscatter phase delay ϕ_k is determined by the shape of the target, geological condition, and weather condition, we can expect $\phi_1 = \phi_2$ in many situations, and hence the interferometric image is obtained as

$$\bar{s}_1 s_2 = |s_1||s_2| e^{\imath(\frac{4\pi(R_1 - R_2)}{\lambda} + \nu)}, \qquad (10)$$

where $\bar{s}_1$ denotes the complex conjugate of s_1 and $\nu := \nu_1 - \nu_2$. The *interferometric phase* $\Theta_{\mathrm{int}} := 4\pi(R_1 - R_2)/\lambda$ can also be expressed, from the simple geometric relation in Fig. 2(a) and the law of cosines, as

$$\Theta_{\mathrm{int}} = \frac{4\pi}{\lambda}\left\{R_1 - \sqrt{R_1^2 + B^2 - 2R_1B\sin(\theta_o - \alpha)}\right\},$$

and its noisy wrapped samples $\Theta_{\mathrm{int}}^W := W(\Theta_{\mathrm{int}} + \nu)$ are observed from (10).

Suppose that we know the height at (x_0, y_0) as H_0 (see Fig. 2(b)). Then we compute the reference phase $\Theta_{\mathrm{ref}} := 4\pi(R_1 - R_2^{H_0})/\lambda$ expressed as

$$\Theta_{\mathrm{ref}} = \frac{4\pi}{\lambda}\left\{R_1 - \sqrt{R_1^2 + B^2 - 2R_1B\sin(\theta_o^{H_0} - \alpha)}\right\}$$

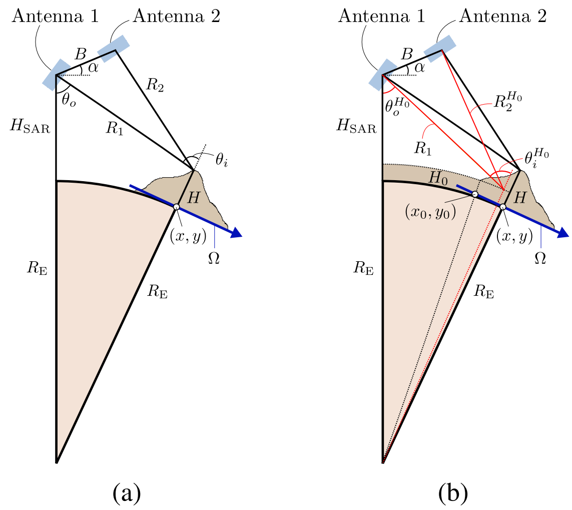


Fig. 2: Outline drawing of terrain height estimation by InSAR. (a) Sectional view for the construction of the interferometric phase. (b) Sectional view for the construction of the reference phase.

s.t. $\cos\theta_o^{H_0} = \frac{R_1^2 + (R_{\mathrm{E}} + H_{\mathrm{SAR}})^2 - (R_{\mathrm{E}} + H_0)^2}{2R_1(R_{\mathrm{E}} + H_{\mathrm{SAR}})}$, which is a virtual interferometric phase assuming that the terrain height is always H_0. Note that the reference phase can be computed because we can compute $\theta_o^{H_0}$ unlike θ_o. Define the 2D unwrapped phase as $\Theta := \Theta_{\mathrm{int}} - \Theta_{\mathrm{ref}}$. To estimate terrain height H, as a refinement of [62, Equation A.2.3], we newly derive the following relation:

$$\Theta \approx \frac{4\pi B\cos(\theta_o^{H_0} - \alpha)(H - H_0)}{\lambda \sin\theta_i^{H_0}\sqrt{R_1^2 + B^2 - 2R_1B\sin(\theta_o^{H_0} - \alpha)}}, \quad (11)$$

where $\theta_i^{H_0}$ in Fig. 2(b) can be computed from $\sin\theta_i^{H_0} = \frac{(R_{\mathrm{E}} + H_{\mathrm{SAR}})\sin\theta_o^{H_0}}{R_{\mathrm{E}} + H_0}$. The wrapped phase $\Theta^W := W(\Theta_{\mathrm{int}} - \Theta_{\mathrm{ref}} + \nu) = W(\Theta_{\mathrm{int}}^W - \Theta_{\mathrm{ref}})$ is obtained from (10) and Θ_{ref}. After reconstructing Θ from Θ^W via 2D phase unwrapping, terrain height H is estimated from (11).

4.2 Parameter Settings of The Proposed Scheme

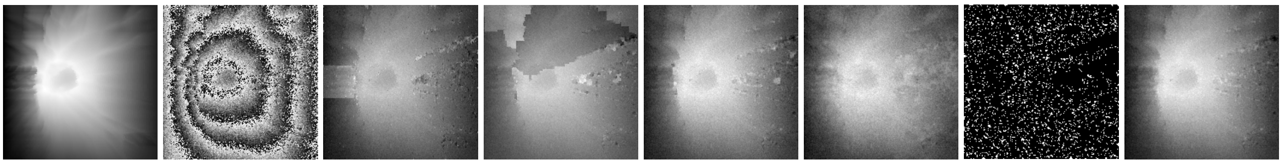
Assume that noisy wrapped samples Θ^W are observed on rectangular grid points $\mathcal{G} := \{(x_i, y_j)\}_{j=0,1,\ldots,m}^{i=0,1,\ldots,n}$ s.t. $x_i - x_{i-1} =: h_x > 0$ $(i = 1,2,\ldots,n)$ and $y_j - y_{j-1} =: h_y > 0$ $(j = 1,2,\ldots,m)$ in $\Omega := [x_0, x_n] \times [y_0, y_m]$.

In DSS, the denoised wrapped samples $\widetilde{\Theta}^W$ on $\mathcal{G}' := \{(x_i', y_j')\}_{j=0,1,\ldots,lm}^{i=0,1,\ldots,ln}$ s.t. $x_0' = x_0$, $x_{ln}' = x_n$, $y_0' = y_0$, $y_{lm}' = y_m$, $x_i' - x_{i-1}' = h_x/l$ $(i = 1,2,\ldots,ln)$, and $y_j' - y_{j-1}' = h_y/l$ $(j = 1,2,\ldots,lm)$ are obtained by using $l = 3$, $\boldsymbol{w}_1 = \boldsymbol{1}$ and $\boldsymbol{w}_2 = \frac{1}{100}\boldsymbol{1}$.

After DSS, we use SPS+ to obtain the smoothest bivariate spline functions $f_{(k)}^* \in \mathcal{S}_4^2(\varDelta_\dagger)$ $(k = 0,1)$, where $\varDelta_\dagger$ is a crisscross partition by diagonally cutting every rectangle $[x_i', x_{i+1}'] \times [y_j', y_{j+1}']$ into four triangles. In SPS+, we set $\epsilon_{(0)}(x,y) = \epsilon_{(1)}(x,y) = 0$ for $(x,y) \in \mathcal{G}_{\mathrm{I}}$ to guarantee

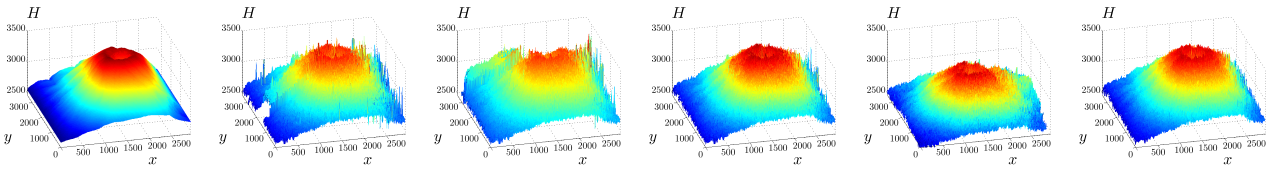
$$W(\theta_{f^*}(x,y)) = \Theta^W(x,y) \quad \text{for all } (x,y) \in \mathcal{G}_{\mathrm{I}}, \quad (12)$$

and we set $\epsilon_{(0)}(x,y) = 0.5 - 0.5|\cos(\widetilde{\Theta}^W(x,y))|$ and $\epsilon_{(1)}(x,y) = 0.5 - 0.5|\sin(\widetilde{\Theta}^W(x,y))|$ for $(x,y) \in \mathcal{G}' \setminus \mathcal{G}_{\mathrm{I}}$ because $\widetilde{\Theta}^W$ is influenced by smoothing effect of DSS and $\mathcal{G}_{\mathrm{II}} \subset \mathcal{G}' \setminus \mathcal{G}_{\mathrm{I}}$.



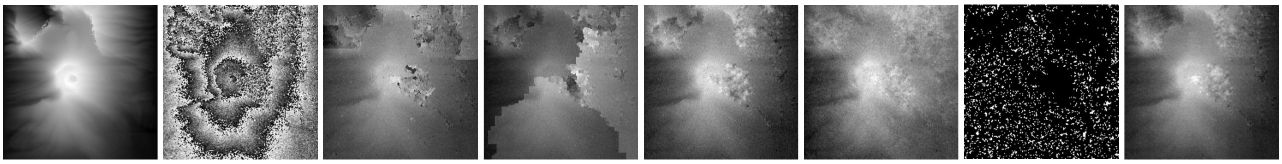
(a) (b) (c) (d) (e) (f) (g) (h)

Fig. 3: Comparison of the proposed 2D phase unwrapping and the existing 2D phase unwrapping (I): (a) unwrapped phase Θ (to be estimated), (b) wrapped phase Θ^W, (c) estimate by BC ($\mathrm{MSE} = 1.7587$), (d) estimate by MST ($\mathrm{MSE} = 8.2192$), (e) estimate by MCF ($\mathrm{MSE} = 0.0974$), (f) estimate by LS ($\mathrm{MSE} = 20.4673$), (g) distribution of Type I (white) and Type II (black), and (h) estimate by the proposed scheme (DSS, SPS+ and APU) ($\mathrm{MSE} = 0.0379$), where MSE is the mean square error of each estimate, i.e, $\mathrm{MSE} := \frac{1}{32761}\sum_{i=0}^{180}\sum_{j=0}^{180}|\Theta_{i,j} - \Theta^*_{i,j}|^2$ (Θ^*: estimate).



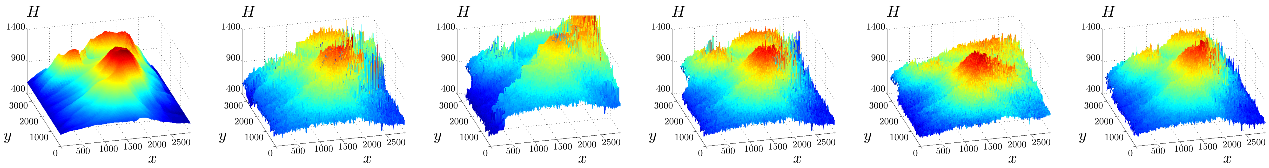
(a) (b) (c) (d) (e) (f)

Fig. 4: Comparison of terrain height estimations based on the proposed 2D phase unwrapping and the existing 2D phase unwrapping (I): (a) test mountain of height H (to be estimated), (b) estimate by BC ($\mathrm{MAE} = 37.6844$), (c) estimate by MST ($\mathrm{MAE} = 87.1949$), (d) estimate by MCF ($\mathrm{MAE} = 26.9321$), (e) estimate by LS ($\mathrm{MAE} = 162.3990$), and (f) estimate by the proposed scheme (DSS, SPS+ and APU) ($\mathrm{MAE} = 23.2882$), where MAE is the mean absolute error of each estimate, i.e., $\mathrm{MAE} := \frac{1}{32761}\sum_{i=0}^{180}\sum_{j=0}^{180}|H_{i,j} - H^*_{i,j}|$ (H^*: estimate).



(a) (b) (c) (d) (e) (f) (g) (h)

Fig. 5: Comparison of the proposed 2D phase unwrapping and the existing 2D phase unwrapping (II): (a) unwrapped phase Θ (to be estimated), (b) wrapped phase Θ^W, (c) estimate by BC ($\mathrm{MSE} = 2.5410$), (d) estimate by MST ($\mathrm{MSE} = 49.4547$), (e) estimate by MCF ($\mathrm{MSE} = 1.4087$), (f) estimate by LS ($\mathrm{MSE} = 5.8364$), (g) distribution of Type I (white) and Type II (black), and (h) estimate by the proposed scheme (DSS, SPS+ and APU) ($\mathrm{MSE} = 0.2011$).



(a) (b) (c) (d) (e) (f)

Fig. 6: Comparison of terrain height estimations based on the proposed 2D phase unwrapping and the existing 2D phase unwrapping (II): (a) test mountain of height H (to be estimated), (b) estimate by BC ($\mathrm{MAE} = 52.1210$), (c) estimate by MST ($\mathrm{MAE} = 210.7460$), (d) estimate by MCF ($\mathrm{MAE} = 41.1130$), (e) estimate by LS ($\mathrm{MAE} = 86.7128$), and (f) estimate by the proposed scheme (DSS, SPS+ and APU) ($\mathrm{MAE} = 30.3923$).

4.3 Numerical Experiments

We demonstrate the effectiveness of the proposed 2D phase unwrapping scheme by terrain height estimation based on (11). Figure 3(a) shows the unwrapped phase Θ generated from a test mountain shown in Fig. 4(a). Here we set the parameters of InSAR system by $\alpha = \pi/6$ [rad], $\lambda = 23.5$ [cm], $B = 500$ [m], $H_{\mathrm{SAR}} = 800$ [km], $R_{\mathrm{E}} = 6371$ [km], $R_1(x_0, y_0) = 1243$ [km], and $H(x_0, y_0) = H_0 = 2530$ [m]. Figure 3(b) depicts the wrapped phase Θ^W on $\mathcal{G} := \{(x_i, y_j)\}_{j=0,1,\ldots,180}^{i=0,1,\ldots,180}$ s.t. $h_x = 16.2$ [m] and $h_y = 19.5$ [m], where additive phase noise ν is generated by [63]. Figures 3(c), 3(d), 3(e), and 3(f) respectively depict the estimates of Θ by *branch cut* (BC)[5], *minimum spanning tree* (MST)[27], *minimum cost flow* (MCF)[28] (all weights are '1'), and *least squares* (LS)[29] (all weights are '1'). Figures 3(g) shows the distribution of samples of Type I and Type II from which we see that samples of Type I distribute sparsely but almost uniformly over Ω. Figure 3(h) depicts the estimate of Θ by the proposed scheme (DSS, SPS+ and APU). Figures 4(b), 4(c), 4(d), 4(e), and 4(f) show the mountains constructed from the results in Fig. 3 and (11). Figures 3 and 4 show that the proposed scheme achieves the best performance compared with the other algorithms visually as well as numerically.

Figure 5(a) shows the unwrapped phase Θ generated from another test mountain in Fig. 6(a). The parameter settings of InSAR system, the proposed scheme, and the other algorithms are same as those used in the first simulation except for $R_1(x_0, y_0) = 1244$ [km] and $H(x_0, y_0) = H_0 = 579$ [m]. Figure 5(b) depicts the noisy wrapped phase Θ^W on $\mathcal{G} := \{(x_i, y_j)\}_{j=0,1\ldots,180}^{i=0,1\ldots,180}$. Figures 5(c), 5(d), 5(e), and 5(f) respectively depict the estimates of Θ by BC, MST, MCF, and LS. Figures 5(g) shows the distribution of samples of Type I and Type II from which we see that samples of Type I of this example also distribute sparsely but almost uniformly over Ω. Figure 5(h) depicts the estimate by the proposed scheme (DSS, SPS+ and APU). Figures 6(b), 6(c), 6(d), 6(e), and 6(f) show the mountains based on the results in Fig. 5 and (11). In this example, the proposed scheme achieves again the best performance compared with the other algorithms.

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