A Smoothness-Aware Phase Unwrapping by Convex Optimization Technique

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Abstract: Two-dimensional (2D) phase unwrapping is a reconstruction problem of a continuous phase, defined over 2D domain, from its wrapped samples, and has been a common key for estimations of some physical information, e.g., the terrain height estimation by the interferometric synthetic aperture radar (InSAR). In this paper, we propose a novel convex cost function for the 2D phase unwrapping. By the newly adding the smoothness measure to the existing cost, we achieve a smooth continuous phase as a minimizer of the proposed convex cost. The simulation of the terrain height estimation by InSAR shows the effectivity of the proposed method.

1 Introduction

Two-dimensional (2D) phase unwrapping [1] is a reconstruction problem of a continuous phase function $\Theta\colon \mathbb{R}^2 \to \mathbb{R}$ from its noisy wrapped samples

$$\Theta^W := W(\Theta + \nu)$$

observed at $(x_i, y_j) \in \mathbb{R}^2$ such that $x_{i+1} - x_i = h_x > 0$ $(i = 0, 1, \ldots, n-1)$ and $y_j - y_{j-1} = h_y > 0$ $(j = 1, 2, \ldots, m)$, where ν is additive noise and $W\colon \mathbb{R} \to (-\pi, \pi]$ is the *wrapping operator* satisfying

$$\forall \theta \in \mathbb{R}\ \exists \eta \in \mathbb{Z}\quad \theta = W(\theta) + 2\pi\theta \ \text{ and } \ W(\theta) \in (-\pi, \pi].$$

The continuous phase $\Theta_{i,j} := \Theta(x_i, y_j)$ is called the *unwrapped phase*, and its noisy wrapped sample $\Theta_{i,j}^W := \Theta^W(x_i, y_j)$ is called the *wrapped phase*. In many applications, the 2D phase unwrapping has been a common key for estimations of some physical information [1], e.g., the terrain height estimation or the landslide identification by the interferometric synthetic aperture radar (InSAR) [2], [3], and the accurate 3D shape measurement by the fringe projection [4].

All existing 2D phase unwrapping algorithms assume that the unwrapped phase difference $\Delta\Theta$ between two neighboring samples is within $\pm\pi$ almost everywhere. Hence most existing algorithms have been designed to suppress a certain cost function $J\colon \mathbb{R}^{(n+1)\times(m+1)} \to \mathbb{R}_+$ measuring the unwrapped phase difference as

$$J(\mathbf{\Theta}) := \sum_{i=0}^{n-1}\sum_{j=0}^{m} w_{i,j}^{x}\left|\Theta_{i+1,j} - \Theta_{i,j} - W\left(\Theta_{i+1,j}^W - \Theta_{i,j}^W\right)\right|^p$$
$$+ \sum_{i=0}^{n}\sum_{j=0}^{m-1} w_{i,j}^{y}\left|\Theta_{i,j+1} - \Theta_{i,j} - W\left(\Theta_{i,j+1}^W - \Theta_{i,j}^W\right)\right|^p, \quad (1)$$

where $\mathbf{\Theta} := (\Theta_{i,j}) \in \mathbb{R}^{(n+1)\times(m+1)}$, $w_{i,j}^x > 0$, $w_{i,j}^y > 0$, and $p > 0$. Specifically, the *branch cut* [2] employs $p \to +0$, the *minimum cost flow* [5] employs $p = 1$, and the *least squares* [6] employs $p = 2$. Such a specification of J is introduced on the basis of a simple property that $\Delta\Theta = W(\Delta\Theta^W)$ holds if $|\Delta\Theta| < \pi$ and $\nu = 0$. Then algorithms finds a minimizer $\mathbf{\Theta}^*$ of J and use it as an estimate of the unwrapped phase. However, if the wrapped phase Θ^W has large additive noise, the minimizer $\mathbf{\Theta}^*$ is also noisy. This is because the cost in (1) is designed on the ideal situation $\nu = 0$.

In this paper, we propose a novel convex cost function for the 2D phase unwrapping. In Section 2, we assume that the shape of the unwrapped phase is smooth. Here the word "smooth" means that the absolute value of the second-order difference, i.e., $|\Delta^2\Theta|$, is small over $\mathbb{R}^2$. Hence the proposed cost function is defined as the sum of (1) and the square of the weighted ℓ_2-norm of the second-order difference, where the values of the weights can be determined by only the wrapped phase information. Then we find a minimizer of the proposed cost by the *alternating direction method of multipliers* [7]. In Section 3, a numerical simulation of the terrain height estimation by InSAR system shows the effectiveness of the proposed method. Finally, in Section 4, we conclude this paper.

2 Recover of Smooth Unwrapped Phase

Assume that the unwrapped phase difference between neighboring samples is within $\pm\pi$ almost everywhere. Then in the nearly noise-free area, we can expect $\Delta\Theta \approx W(\Delta\Theta^W)$. However, in the following situation, there is a possibility that we encounter $\Delta\Theta \not\approx W(\Delta\Theta^W)$.

- When $|\Delta\Theta|$ is close to π, $W(\Delta\Theta^W)$ can be easily changed from $\Delta\Theta$ even by small additive noise, e.g., if $\Delta\Theta = 0.95\pi$ and $\Delta\nu = 0.1$, then $W(\Delta\Theta^W) = W(\Delta\Theta + \Delta\nu) = W(1.05\pi) = -0.95\pi \not\approx \Delta\Theta$.

On the other hand, in noisy area, the value of $W(\Delta\Theta^W)$ is unreliable due to ν, and the following holds (see [1]).

- In noisy area there are many rectangles $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ having the so-called *residue*. By computing

$$r_{i,j} := W\left(\Theta_{i,j+1}^W - \Theta_{i,j}^W\right) + W\left(\Theta_{i+1,j+1}^W - \Theta_{i,j+1}^W\right) + W\left(\Theta_{i+1,j}^W - \Theta_{i+1,j+1}^W\right) + W\left(\Theta_{i,j}^W - \Theta_{i+1,j}^W\right) = \begin{cases} 0 & \text{(not residue)}; \\ \pm 2\pi & \text{(positive/negative residue)}, \end{cases}$$

  we find out if each rectangle has a residue or not.

In order to reconstruct the unwrapped phase in the area where $W(\Delta\Theta^W)$ is unreliable, we assume that the shape of the unwrapped phase is smooth. As a result, we solve the following convex optimization problem:

Find $\mathbf{\Theta}^* \in \mathbb{R}^{(n+1)\times(m+1)}$ minimizing

$$J(\mathbf{\Theta}) := \sum_{i=0}^{n-1}\sum_{j=0}^{m} w_{i,j}^{x}\left|\Theta_{i+1,j} - \Theta_{i,j} - W\left(\Theta_{i+1,j}^{W} - \Theta_{i,j}^{W}\right)\right|$$
$$+\sum_{i=0}^{n}\sum_{j=0}^{m-1} w_{i,j}^{y}\left|\Theta_{i,j+1} - \Theta_{i,j} - W\left(\Theta_{i,j+1}^{W} - \Theta_{i,j}^{W}\right)\right|$$
$$+\sum_{i=0}^{n-2}\sum_{j=0}^{m} w_{i,j}^{xx}\left|\Theta_{i+2,j} - 2\Theta_{i+1,j} + \Theta_{i,j}\right|^2$$
$$+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} w_{i,j}^{xy}\left|\Theta_{i+1,j+1} - \Theta_{i+1,j} - \Theta_{i,j+1} + \Theta_{i,j}\right|^2$$
$$+\sum_{i=0}^{n}\sum_{j=0}^{m-2} w_{i,j}^{yy}\left|\Theta_{i,j+2} - 2\Theta_{i,j+1} + \Theta_{i,j}\right|^2, \quad (2)$$

where the values of two weights $w_{i,j}^{x} > 0$ and $w_{i,j}^{y} > 0$ decrease with increasing values of $|W(\Theta_{i+1,j}^{W} - \Theta_{i,j}^{W})|$ and $|W(\Theta_{i,j+1}^{W} - \Theta_{i,j}^{W})|$ respectively, and the values of the other weights $w_{i,j}^{xx} > 0$, $w_{i,j}^{xy} > 0$ and $w_{i,j}^{yy} > 0$ increase with increasing number of the residues in the neighborhood of a rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$. Since the cost function J in (2) is convex, we obtain $\mathbf{\Theta}^*$ by the alternating direction method of multipliers [7].

3 Terrain Height Estimation by InSAR

The interferometric synthetic aperture radar (InSAR) [2], [3] is an imaging technique allowing highly accurate measurements of the surface topography in all weather conditions. In the InSAR system [8], the terrain height H is estimated from the unwrapped phase Θ through

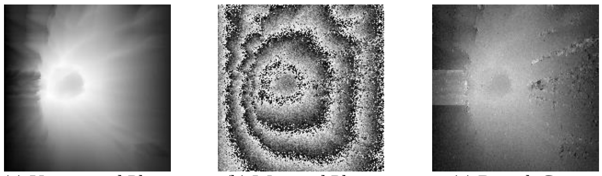
$$\Theta := \Theta_{\text{int}} - \Theta_{\text{ref}} \approx \frac{4\pi B\cos(\theta_o - \alpha)}{\lambda R \sin\theta_i}(H - H_0), \quad (3)$$

where Θ_{int} is the *interferometric phase*, Θ_{ref} is the *reference phase*, H_0 is the height of the reference plane, λ is the wavelength of the transmitted signal, R is the distance from the antenna to the target, θ_o is the off-nadir angle, θ_i is the incident angle, and α and B are the parameters of the InSAR system.

Figure 1(a) shows the true 2D unwrapped phase Θ generated from a virtual mountain in Figure 2(a), and Figure 1(b) shows the observed noisy wrapped samples $\Theta_{i,j}^{W}$. Figures 1(c), 1(d), 1(e), and 1(f) respectively depict the estimates by the branch cut [2], the minimum cost flow [5], the least squares [6], and the propose method. Figures 2(b), 2(c), 2(d), and 2(e) show the terrain height computed from (3). From Figures 1 and 2, we find out that the proposed method gives the best performance.

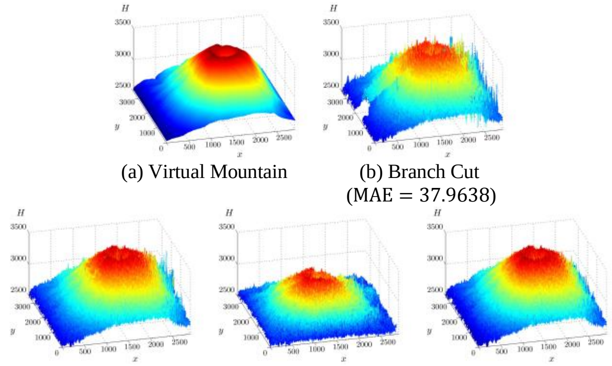
4 Conclusion

In this paper, we proposed a novel convex cost for the 2D phase unwrapping. By newly adding the weighted sum of the square of the second-order difference as the cost, a minimizer of the proposed convex cost can be expect to be smooth. The simulation of the terrain height estimation by InSAR showed the effectivity of the proposed cost function. Finally, we remark that the proposed method can be utilized as an effective preprocessing of the 2D *algebraic phase unwrapping* [9] to satisfy the wrapped phase condition.



(a) Unwrapped Phase (b) Wrapped Phase (c) Branch Cut (MSE = 1.7873)
(d) Minimum Cost Flow (MSE = 0.0971) (e) Least Squares (MSE = 36.6905) (f) Proposed Method (MSE = 0.0275)

Figure 1: Estimates of the unwrapped phase and their mean square errors.



(a) Virtual Mountain (b) Branch Cut (MAE = 37.9638)
(c) Minimum Cost Flow (MAE = 27.1063) (d) Least Squares (MAE = 212.8230) (e) Proposed Method (MAE = 25.4376)

Figure 2: Estimates of the terrain height and their mean absolute errors.

References

[1] D. C. Ghiglia and M. D. Pritt, *Two-Dimensional Phase Unwrapping: Theory, Algorithms, and Software*. New York, NY: Wiley, 1998.

[2] R. M. Goldstein, H. A. Zebker, and C. L. Werner, "Satellite radar interferometry: Two-dimensional phase unwrapping," *Radio Science*, vol. 23, no. 4, pp. 713–720, Jul./Aug. 1988.

[3] C. Colesanti and J. Wasowski, "Investigating landslides with space-borne synthetic aperture radar (SAR) interferometry," *Engineering Geology*, vol. 88, no. 3-4, pp. 173–199, Dec. 2006.

[4] H. Zhao, W. Chen, and Y. Tan, "Phase-unwrapping algorithm for the measurement of three-dimensional object shapes," *Applied Optics*, vol. 33, no. 20, pp. 4497–4500, Jul. 1994.

[5] M. Costantini, "A novel phase unwrapping method based on network programming," *IEEE Transactions on Geoscience and Remote Sensing*, vol. 36, no. 3, pp. 813–821, May 1998.

[6] D. C. Ghiglia and L. A. Romero, "Robust two-dimensional weighted and unweighted phase unwrapping that uses fast transforms and iterative methods," *Journal of the Optical Society of America A: Optics, Image Science, and Vision*, vol. 11, no. 1, pp. 107–117, 1994.

[7] D. Gabay and B. Mercier, "A dual algorithm for the solution of nonlinear variational problems via finite element approximation," *Computers & Mathematics with Applications*, vol. 2, no. 1, pp. 17–40, 1976.

[8] R. F. Hanssen, *Radar Interferometry: Data Interpretation and Error Analysis*. Dordrecht, The Netherlands: Kluwer 2001

[9] D. Kitahara and I. Yamada, "Algebraic phase unwrapping along the real axis: extensions and stabilizations," *Multidimensional Systems and Signal Processing*, vol. 26, no. 1, pp. 3–45, Jan 2015.