A Preprocessing Using Convex Optimization for Algebraic Two-Dimensional Phase Unwrapping

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Abstract Two-dimensional (2D) phase unwrapping is a reconstruction problem of a 2D continuous phase from its wrapped samples. In our previous work, we presented a two-step phase unwrapping algorithm which first constructs a complex function whose real and imaginary parts are smooth piecewise polynomials having no common zero, then estimates the unwrapped phase by applying the algebraic phase unwrapping. In this paper, we propose a preprocessing of the above algorithm for avoiding the appearance of zeros of the complex function in the first step. The proposed preprocessing is implemented by a convex optimization and resampling, and its effectiveness is shown in a terrain height estimation by the interferometric synthetic aperture radar.

1 INTRODUCTION

Two-dimensional (2D) phase unwrapping [1], [2] is a reconstruction problem of an unknown continuous phase function $\Theta : \Omega \to \mathbb{R}$ defined in a simply connected closed region $\Omega \subset \mathbb{R}^2$, from its finite noisy wrapped samples

$$\Theta^W(x,y) := W(\Theta(x,y) + \nu(x,y)) \in (-\pi, \pi]$$

observed at $(x,y) \in \mathcal{G} \subset \Omega$, where ν is additive noise, $\mathcal{G}$ stands for the set of observation points, and $W : \mathbb{R} \to (-\pi, \pi]$ is the wrapping operator satisfying

$$\forall x \in \mathbb{R}\ \exists \eta \in \mathbb{Z} \quad x = 2\pi\eta + W(x) \text{ and } W(x) \in (-\pi, \pi].$$

The continuous phase Θ is called the *unwrapped phase* and its wrapped sample Θ^W is called the *wrapped phase*. In many signal and image processing, the 2D phase unwrapping has been a common key for estimations of some physical information [1], [2], for example, the terrain height estimation or the landslide identification by the interferometric synthetic aperture radar (InSAR) [3], [4], the seafloor depth estimation by the interferometric synthetic aperture sonar [5], the accurate 3D shape measurement by the fringe projection [6] or x-ray [7], and the water/fat separation in the magnetic resonance imaging [8].

All existing phase unwrapping algorithms assume that the unwrapped phase difference between two neighboring samples is within $\pm\pi$ almost everywhere. Hence most algorithms construct a cost function J about the unwrapped phase difference as

$$J(\boldsymbol{\Theta}) := \sum_i J_i(\Delta\Theta_i) \quad \text{s.t. } \boldsymbol{\Theta} := (\Theta(x,y))_{(x,y)\in\mathcal{G}}, \quad (1)$$

where $J_i : \mathbb{R} \to \mathbb{R}_+$ is designed to achieve 0 at $\Delta\Theta_i = W(\Delta\Theta_i^W)$, $\Delta\Theta_i$ and $\Delta\Theta_i^W$ are respectively the unwrapped and wrapped phase differences between ith pair of neighboring samples. Such a design of J_i is based on a simple observation that $\Delta\Theta_i = W(\Delta\Theta_i^W)$ holds if $|\Delta\Theta_i| < \pi$ and $\nu = 0$. Then the algorithms use a minimizer $\boldsymbol{\Theta}^*$ of J as an estimate of the unwrapped phase.

Moreover, the existing algorithms can be divided into two types. Major algorithms [3], [9], [10] find a minimizer of (1) under the condition

$$\forall (x,y) \in \mathcal{G}\ \exists \eta \in \mathbb{Z} \quad \Theta(x,y) = \Theta^W(x,y) + 2\pi\eta. \quad (2)$$

The above optimization problem is combinatorial and intractable due to constraint (2). Therefore the algorithms in this type, find at first closed loops, having the so-called *residues*, where there is at least one $\Delta\Theta_i$ differing from $W(\Delta\Theta_i^W)$. After finding the residues, the algorithms construct the edge by connecting the residues. Here the set of indices of the edge is denoted by E. Then, we can construct $\boldsymbol{\Theta}$ satisfying that $\Delta\Theta_i = W(\Delta\Theta_i^W)$ if $i \notin E$, and $\Delta\Theta_i \neq W(\Delta\Theta_i^W)$ if $i \in E$. As a result, (1) is expressed as

$$\sum_{i\in E} J_i(\Delta\Theta_i) + \sum_{i\notin E} J_i(\Delta\Theta_i) = \sum_{i\in E} J_i(\Delta\Theta_i) =: \widehat{J}(E), \quad (3)$$

and hence the algorithms find an optimal edge E^* minimizing (3). In this paper, we call the algorithms in this type the *network-flow methods* along [2], [10], because these algorithms find an optimal edge E^* by using the idea of the flow network in the graph theory.

In this approach, if the observed wrapped phase has only small additive noise and the true unwrapped phase difference is enough small compared with the sampling interval, we can obtain an optimal edge E^* and very good estimate $\boldsymbol{\Theta}(E^*)$. However, otherwise, not only constraint (2) is violated due to the additive noise, but also we cannot find an optimal edge E^* in many cases due to the increase of the number of the residues and the NP-hardness of the combinatorial optimization problem [10].

The algorithms in other type [11], [12] find a minimizer of (1) without constraint (2). In this approach, if the cost function is convex, we can find a minimizer $\boldsymbol{\Theta}^*$, and the computation time does not depend on the number of the residues but the size of vector $\boldsymbol{\Theta}$. Therefore in case that the observed wrapped phase has large additive noise and many residues, the algorithms in this type are effective. However there is no guarantee about the error between $W(\Theta^*(x,y))$ and $\Theta^W(x,y)$ at each $(x,y) \in \mathcal{G}$, which ordinary destroys the sharp edges of the true unwrapped phase.

In [13], we proposed a completely different phase unwrapping algorithm which is composed of two steps. First, the proposed algorithm constructs a twice differentiable complex function $f := f_{(0)} + \imath f_{(1)} = |f|e^{\imath\theta_f}$, where $f \neq 0$ over Ω, and $f_{(0)}$ and $f_{(1)}$ are twice continuously differentiable spline functions respectively approximating $\cos(\Theta)$ and $\sin(\Theta)$. Second a continuous phase function $\theta_f \in C^2(\Omega)$ of f is exactly computed by the algebraic phase unwrapping [16]–[18], and θ_f is used as an estimate. However in case of the wrapped phase has many residues, f obtained in the first step often has many zeros in Ω, which results in the failure of the construction of θ_f in the second step.

Therefore, in this paper, we propose a preprocessing of the phase unwrapping [13] to avoid the generation of zeros of f. The first step of this preprocessing is given in Sect. 3.1 where we find a minimizer $\mathbf{\Theta}^*$ of a newly defined convex cost function without constraint (2). If the unwrapped and wrapped phase differences are respectively denoted by $\Delta\Theta_{i,j}$ and $\Delta\Theta_{i,j}^W$, the cost function is defined to encourage $\Delta\Theta_{i,j}^* \approx W(\Delta\Theta_{i,j}^W)$ if $|W(\Delta\Theta_{i,j}^W)|$ is small, and to promote the smoothness of $(\Theta_{i,j}^*)$ otherwise. The second step of the preprocessing is given in Sect. 3.2 where we produce a virtual wrapped phase $\widehat{\Theta}^W$, over finer grid than $\mathcal{G}$, based on $\mathbf{\Theta}^*$ and Θ^W. Finally, we construct θ_f from this virtual wrapped phase by using the phase unwrapping algorithm [13]. In Sect. 4, a numerical simulation of a terrain height estimation by InSAR is given, which shows the effectiveness of the proposed preprocessing and phase unwrapping algorithm.

Notation Let $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_{++}$ and $\mathbb{C}$ denote respectively the set of all integers, non-negative integers, real numbers, non-negative real numbers, positive real numbers, and complex numbers. We use $\imath \in \mathbb{C}$ to denote the imaginary unit satisfying $\imath^2 = -1$, and $i \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_+$ are used as the indices. For $\rho \in \mathbb{Z}_+$, $C^\rho(\Omega)$ stands for the set of all ρ-times continuously differentiable functions over the interior of a simply connected closed region $\Omega \subset \mathbb{R}^2$. A boldface letter denotes a vector or a matrix depending on the situation. For any vector $\boldsymbol{x} \in \mathbb{R}^n$ and diagonal matrix $\boldsymbol{X} \in \mathbb{R}^{n\times n}$, $[\boldsymbol{x}]_i$ and $[\boldsymbol{X}]_i$ respectively denote the ith component of $\boldsymbol{x}$ and (i,i)-th entry of $\boldsymbol{X}$. For any $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{w} \in \mathbb{R}_{++}^n$ and $p \geq 1$, weighted ℓ_p-norm is defined as $\|\boldsymbol{x}\|_{p,\boldsymbol{w}} := \sqrt[p]{\sum_{i=1}^n [\boldsymbol{w}]_i \cdot |[\boldsymbol{x}]_i|^p}$.

2 Existing 2D Phase Unwrapping Algorithms

In what follows, for simplicity, assume that the wrapped samples are observed at regular rectangular grid points $\mathcal{G} := \{(x_i, y_j) \mid i = 0, 1, \ldots, n \text{ and } j = 0, 1, \ldots, m\}$ on a simply connected closed region $\Omega := [x_0, x_n] \times [y_0, y_m]$ s.t. $x_0 < x_1 < \cdots < x_n$, $y_0 < y_1 < \cdots < y_m$, $x_{i+1} - x_i = h_x$ $(i = 0, 1, \ldots, n-1)$ and $y_{j+1} - y_j = h_y$ $(j = 0, 1, \ldots, m-1)$. Moreover define $\Theta_{i,j} := \Theta(x_i, y_j)$, $\Theta_{i,j}^W := \Theta^W(x_i, y_j)$ and $\mathbf{\Theta} := \mathrm{vec}(\Theta_{i,j}) \in \mathbb{R}^{(n+1)(m+1)}$. In this case, $J : \mathbb{R}^{(n+1)(m+1)} \to \mathbb{R}_+$ in (1) is expressed as

$$\sum_{i=0}^{n-1}\sum_{j=0}^{m} J_{i,j}^x(\Theta_{i+1,j} - \Theta_{i,j}) + \sum_{i=0}^{n}\sum_{j=0}^{m-1} J_{i,j}^y(\Theta_{i,j+1} - \Theta_{i,j}), \tag{4}$$

where $J_{i,j}^x : \mathbb{R} \to \mathbb{R}_+$ and $J_{i,j}^y : \mathbb{R} \to \mathbb{R}_+$ generally reach a minimum 0 at $\Theta_{i+1,j} - \Theta_{i,j} = W(\Theta_{i+1,j}^W - \Theta_{i,j}^W)$ and $\Theta_{i,j+1} - \Theta_{i,j} = W(\Theta_{i,j+1}^W - \Theta_{i,j}^W)$ respectively.

The network-flow methods [3], [9], [10] find a minimizer of (4) under the constraint

$$\forall i,j \quad \exists \eta_{i,j} \in \mathbb{Z} \quad \Theta_{i,j} = \Theta_{i,j}^W + 2\pi\eta_{i,j}. \tag{5}$$

For solving this combinatorial problem, the network-flow methods, at first, identify rectangles $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ satisfying

$$\begin{aligned} &W\left(\Theta_{i,j+1}^W - \Theta_{i,j}^W\right) + W\left(\Theta_{i+1,j+1}^W - \Theta_{i,j+1}^W\right) \\ &\quad \neq W\left(\Theta_{i+1,j}^W - \Theta_{i,j}^W\right) + W\left(\Theta_{i+1,j+1}^W - \Theta_{i+1,j}^W\right). \end{aligned}$$

These rectangles are called the residues, and by computing

$$\begin{aligned} r_{i,j} &:= \frac{1}{2\pi}\Big\{W\left(\Theta_{i,j+1}^W - \Theta_{i,j}^W\right) + W\left(\Theta_{i+1,j+1}^W - \Theta_{i,j+1}^W\right) \\ &\qquad - W\left(\Theta_{i+1,j+1}^W - \Theta_{i+1,j}^W\right) - W\left(\Theta_{i+1,j}^W - \Theta_{i,j}^W\right)\Big\} \\ &= \begin{cases} 0 & \text{(not residue)}, \\ +1 & \text{(positive residue)}, \\ -1 & \text{(negative residue)}, \end{cases} \end{aligned}$$

we find out whether each rectangle is a positive/negative residue or not (see Fig. 1(a)). After identifying the residues, the network-flow methods create the set E of edges connecting the positive and negative residues with correspondence of the numbers of positive and negative residues, and then we can construct $\mathbf{\Theta}(E)$ whose differences between two neighboring points exceed $\pm\pi$ in only the area striding over the edge (see Fig. 1(b)). We express E as $E := (E_x, E_y)$, where E_x and E_y are the sets of indices satisfying

$$\left.\begin{aligned} (i,j) \in E_x &\Leftrightarrow (\Theta_{i+1,j}(E) - \Theta_{i,j}(E)) \neq W\left(\Theta_{i+1,j}^W - \Theta_{i,j}^W\right) \\ (i,j) \in E_y &\Leftrightarrow (\Theta_{i,j+1}(E) - \Theta_{i,j}(E)) \neq W\left(\Theta_{i,j+1}^W - \Theta_{i,j}^W\right) \end{aligned}\right\}.$$

As a result, if

$$J_{i,j}^x\left(W\left(\Theta_{i+1,j}^W - \Theta_{i,j}^W\right)\right) = J_{i,j}^y\left(W\left(\Theta_{i,j+1}^W - \Theta_{i,j}^W\right)\right) = 0$$

for all i and j, then the cost in (4) is expressed as

$$\begin{aligned} J(\mathbf{\Theta}(E)) &= \sum_{(i,j)\in E_x} J_{i,j}^x(\Theta_{i+1,j}(E) - \Theta_{i,j}(E)) \\ &\quad + \sum_{(i,j)\in E_y} J_{i,j}^y(\Theta_{i,j+1}(E) - \Theta_{i,j}(E)) = \widehat{J}(E). \end{aligned}$$

Therefore the network-flow methods find an optimal edge $E^* = (E_x^*, E_y^*)$ minimizing (6) and then construct $\mathbf{\Theta}^*$ as the estimate of the unwrapped phase.

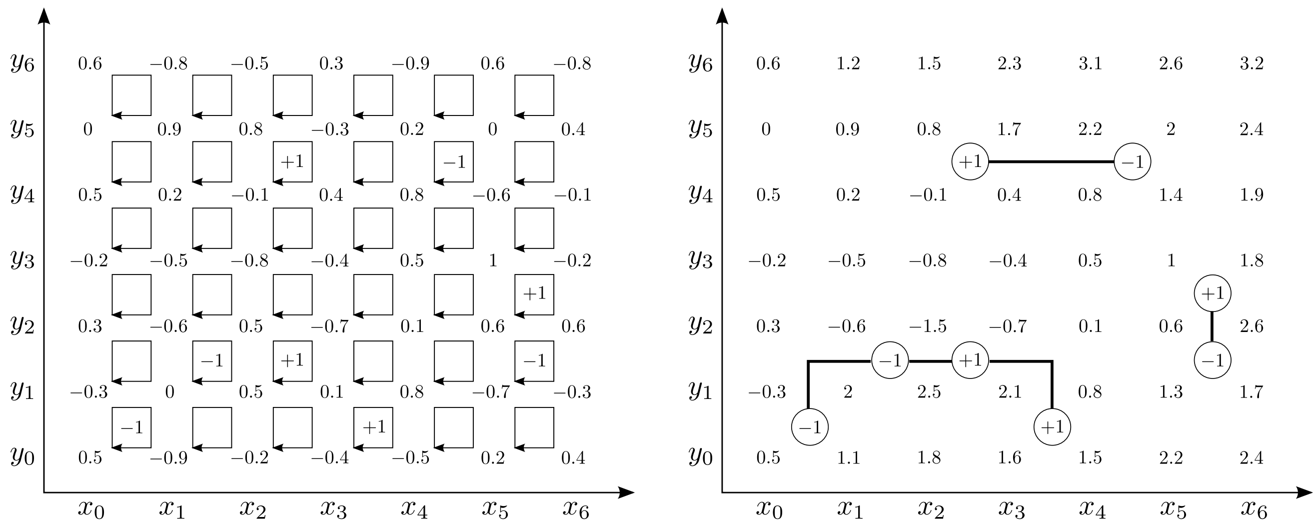
The other algorithms [11], [12] directly find minimizer $\mathbf{\Theta}(E^*)$ of (4) without the constraint in (5). In this section, we explain three major algorithms, the *branch cut*, the *minimum cost flow*, which are in the class of the network-flow methods, and the *minimum ℓ_p-norm*, which is not.

2.1 Branch Cut

The branch cut (BC) algorithm was established by Goldstein et al. [3] and minimizes ℓ_0-norm under constraint (5). As a result, the BC finds $E^* = (E_x^*, E_y^*)$ minimizing

$$|E_x| + |E_y|.$$

Since this problem is NP-hard [10], there are no methods which can solve this problem in polynomial time. The BC constructs E by connecting repeatedly the nearest residues without checking whether the residues have already been connected with other residues. Therefore the same residues are connected many times. As a result, in areas of dense residues, the BC makes many extra edges and results in awful deterioration of the estimate accuracy [10].



(a) $\Theta^W_{i,j}$ $[\mathrm{rad}/\pi]$ and residues $r_{i,j}$

(b) $\Theta_{i,j}(E)$ $[\mathrm{rad}/\pi]$ and edges

Figure 1: An example of the unwrapped phase estimated from the wrapped phase by the network-flow method. The edges $E_x := \{(0,1),(3,1),(5,2)\}$ and $E_y := \{(1,1),(2,1),(3,1),(3,4),(4,4)\}$ are obtained by connecting the residues.

2.2 Minimum Cost Flow

The minimum cost flow (MCF) algorithm was established by Costantini [9] and minimizes weighted ℓ_1-norm under constraint (5). As a result, the MCF finds $E^* = (E_x^*, E_y^*)$ minimizing

$$\sum_{(i,j)\in E_x} w^x_{i,j} \left|\Theta_{i+1,j}(E) - \Theta_{i,j}(E) - W\left(\Theta^W_{i+1,j} - \Theta^W_{i,j}\right)\right|$$
$$+ \sum_{(i,j)\in E_y} w^y_{i,j} \left|\Theta_{i,j+1}(E) - \Theta_{i,j}(E) - W\left(\Theta^W_{i,j+1} - \Theta^W_{i,j}\right)\right|,$$

where $w^x_{i,j} > 0$ and $w^y_{i,j} > 0$ are weights. Then by considering a positive residue as a *supply*, a negative residue as a *demand*, and the weight as the cost that must be paid per flow, the above optimization problem is found to be same as a *minimum cost integer-flow problem*, i.e., an optimal integer-flow is correspond to an optimal edge E^*. As a result, $\mathbf{\Theta}(E^*)$ is computed by solving this minimum cost integer-flow problem by the algorithms in the graph theory.

2.3 Minimum ℓ_p-Norm

Differently from the previous network-flow methods, the minimum ℓ_p-norm (MLN) algorithm was established by Ghiglia and Romero [12] as the generalized version of the minimum ℓ_2-norm algorithm (so-called the *least squares method*) [11]. The MLN finds $\mathbf{\Theta}^*$ minimizing

$$\sum_{i=0}^{n-1}\sum_{j=0}^{m} w^x_{i,j} \left|\Theta_{i+1,j} - \Theta_{i,j} - W\left(\Theta^W_{i+1,j} - \Theta^W_{i,j}\right)\right|^p$$
$$+ \sum_{i=0}^{n}\sum_{j=0}^{m-1} w^y_{i,j} \left|\Theta_{i,j+1} - \Theta_{i,j} - W\left(\Theta^W_{i,j+1} - \Theta^W_{i,j}\right)\right|^p$$

without the constraint in (5), where $w^x_{i,j}, w^y_{i,j} > 0$ and $p > 0$. If $p \geq 1$, the cost function is convex, and hence we can obtain a minimizer $\mathbf{\Theta}^*$. In particular, for $p = 2$ and $w^x_{i,j} = w^y_{i,j} = 1$, i.e., the non-weighted least squares case, $\mathbf{\Theta}^*$ is computed very efficiently by using FFT [11]. However since there is no guarantee on the possible gap between $W(\Theta^*_{i,j})$ and $\Theta^W_{i,j}$, the estimate by the MLN is generally too smooth, i.e., $\Delta\Theta^*_{i,j}$ tends to be small over the whole area of Ω.

3 Algebraic Recovery of 2D Unwrapped Phase

In our previous work [13], we proposed a completely different algebraic approach to the 2D phase unwrapping problem. We estimate Θ by a phase function θ_f of a twice differentiable complex function $f := f_{(0)} + \imath f_{(1)} = |f|e^{\imath\theta_f}$, where $f_{(0)} \in C^2(\Omega)$ and $f_{(1)} \in C^2(\Omega)$ respectively approximate $\cos(\Theta)$ and $\sin(\Theta)$. In the spirit of functional data analysis [14], [15], we use the smoothest spline function, which is consistent with given wrapped phase information, as f in order to obtain the smooth phase function θ_f. The proposed approach is composed of the following two steps.

Step 1: Find $f^*_{(k)} \in \mathcal{S}^2_4(\Delta) \subset C^2(\Omega)$ $(k = 0, 1)$ minimizing

$$\iint_\Omega \left[\left|\frac{\partial^2 f_{(k)}}{\partial x^2}\right|^2 + 2\left|\frac{\partial^2 f_{(k)}}{\partial x \partial y}\right|^2 + \left|\frac{\partial^2 f_{(k)}}{\partial y^2}\right|^2\right] dxdy$$

subject to

$$\left.\begin{array}{l} -\epsilon^{(0)}_{i,j} \leq f_{(0)}(x_i, y_j) - \cos([\Theta^W_{i,j}]) \leq \epsilon^{(0)}_{i,j} \\ -\epsilon^{(1)}_{i,j} \leq f_{(1)}(x_i, y_j) - \sin([\Theta^W_{i,j}]) \leq \epsilon^{(1)}_{i,j} \end{array}\right\}$$

for each $(x_i, y_j) \in \mathcal{G}$, where $\mathcal{S}^2_4(\Delta)$ denotes the set of all bivariate spline functions of degree 4 and smoothness 2, and $\epsilon^{(0)}_{i,j} \geq 0$ and $\epsilon^{(1)}_{i,j} \geq 0$ are acceptable errors for the wrapped phase information.

Step 2: Compute a phase function of $f^* := f^*_{(0)} + \imath f^*_{(1)} = |f^*|e^{\imath\theta_{f^*}}$ for any point of interest $(x, y) \in \Omega$.

Step 1 is implemented by solving a convex optimization problem about the coefficients of the spline function [13]. Then if f^* does not have zeros over Ω, a twice continuously differentiable function $\theta_{f^*} \in C^2(\Omega)$ is defined as

$$\theta_{f^*}(x, y) := \theta_{f^*}(x_0, y_0)$$
$$+ \int_a^b \Im\left[\frac{\left(f^*_{(0)}(\Upsilon(t))\right)' + \imath\left(f^*_{(1)}(\Upsilon(t))\right)'}{f^*_{(0)}(\Upsilon(t)) + \imath f^*_{(1)}(\Upsilon(t))}\right] dt,$$

where $\Upsilon : [a,b] \to \Omega$ is any piecewise C^1 path satisfying $\Upsilon(a) = (x_0, y_0)$ and $\Upsilon(b) = (x, y)$, and $\Im(\cdot)$ denotes the imaginary part of the argument. In Step 2, this integral is computed by the algebraic phase unwrapping [16]–[18].

However in case where the observed wrapped phase has many residues, f^* obtained in Step 1 of the algorithm [13]. also tends to have many zeros over Ω, which results in the path dependence of the obtained unwrapped phase in Step 2. Therefore we need resampling to avoid the generation of zeros of f^*. By observing the fact seen, e.g., in the MLP in Sect. 2.3, that we can obtain an over-smooth estimate by minimizing of a convex cost function without (5), we propose the following two-step resampling method.

Step A: Reconstruct the rough geometry of a unknown continuous function Θ by finding a minimizer $\boldsymbol{\Theta}^*$ of a cost function without the constraint in (5).

Step B: Produce the virtual wrapped phase $\widehat{\Theta}^W(x'_i, y'_j)$, based on $\boldsymbol{\Theta}^*$ and Θ^W, at $(x'_i, y'_j) \in \mathcal{G}'$, where $\mathcal{G}' \supset \mathcal{G}$ is the set of regular rectangular grid points whose grid interval is finer than $\mathcal{G}$.

3.1 Convex Optimization Problem in Step A

Assume that the unwrapped phase differences between almost all pairs of neighboring samples are within $\pm\pi$, and the observed wrapped phase has small additive noise almost everywhere. Then, at many points on Ω, we can expect $\Delta\Theta_{i,j} \approx W(\Delta\Theta^W_{i,j})$. However, in the following situations, there is a possibility that we encounter $\Delta\Theta_{i,j} \not\approx W(\Delta\Theta^W_{i,j})$.

- When $|\Delta\Theta_{i,j}|$ is close to π, $W(\Delta\Theta^W_{i,j})$ can easily different from $\Delta\Theta_{i,j}$ even by small additive noise, e.g., if $\Delta\Theta_{i,j} = 0.95\pi$ and $\Delta\nu_{i,j} = 0.1\pi$, then $W(\Delta\Theta^W_{i,j}) = W(\Delta\Theta_{i,j} + \Delta\nu_{i,j}) = W(1.05\pi) = -0.95\pi \not\approx \Delta\Theta_{i,j}$.
- In the neighborhood of the residues, there is at least one $\Delta\Theta_{i,j} \not\approx W(\Delta\Theta^W_{i,j})$ like Fig. 1(b).

In the above areas, we try to construct smooth $\boldsymbol{\Theta}$ in disregard of $W(\Delta\Theta^W_{i,j})$. Here the word "smooth" means that the absolute value of the second order discrete gradient is small.

As a result, we solve the following convex optimization problem in Step A: Find $\boldsymbol{\Theta}^* \in \mathbb{R}^{(n+1)(m+1)}$ minimizing

$$
\begin{aligned}
&\sum_{i=0}^{n-1}\sum_{j=0}^{m} w^x_{i,j}\left|\Theta_{i+1,j} - \Theta_{i,j} - W\left(\Theta^W_{i+1,j} - \Theta^W_{i,j}\right)\right| \\
&\quad + \sum_{i=0}^{n}\sum_{j=0}^{m-1} w^y_{i,j}\left|\Theta_{i,j+1} - \Theta_{i,j} - W\left(\Theta^W_{i,j+1} - \Theta^W_{i,j}\right)\right| \\
&\quad + \sum_{i=0}^{n-2}\sum_{j=0}^{m} w^{xx}_{i,j}\left|\Theta_{i+2,j} - 2\Theta_{i+1,j} + \Theta_{i,j}\right|^2 \\
&\quad + \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} w^{xy}_{i,j}\left|\Theta_{i+1,j+1} - \Theta_{i+1,j} - \Theta_{i,j+1} + \Theta_{i,j}\right|^2 \\
&\quad + \sum_{i=0}^{n}\sum_{j=0}^{m-2} w^{yy}_{i,j}\left|\Theta_{i,j+2} - 2\Theta_{i,j+1} + \Theta_{i,j}\right|^2 \\
&= \|\boldsymbol{D}_x\boldsymbol{\Theta} - \boldsymbol{\delta}_x\|_{1,\boldsymbol{w}_x} + \|\boldsymbol{D}_y\boldsymbol{\Theta} - \boldsymbol{\delta}_y\|_{1,\boldsymbol{w}_y} \\
&\qquad + \|\boldsymbol{D}_{xx}\boldsymbol{\Theta}\|^2_{2,\boldsymbol{w}_{xx}} + \|\boldsymbol{D}_{xy}\boldsymbol{\Theta}\|^2_{2,\boldsymbol{w}_{xy}} + \|\boldsymbol{D}_{yy}\boldsymbol{\Theta}\|^2_{2,\boldsymbol{w}_{yy}}
\end{aligned}
$$

where two weights $w^x_{i,j}$ and $w^y_{i,j}$ decrease with the increasing $|W(\Delta\Theta^W_{i,j})|$ and are vectorized as $\boldsymbol{w}_x := \mathrm{vec}(w^x_{i,j})$ and $\boldsymbol{w}_y := \mathrm{vec}(w^y_{i,j})$, the other weights $w^{xx}_{i,j}$, $w^{xy}_{i,j}$, and $w^{yy}_{i,j}$ increase with the increasing the number of the residues in the neighborhood of a rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ and are vectorized as $\boldsymbol{w}_{xx} := \mathrm{vec}(w^{xx}_{i,j})$, $\boldsymbol{w}_{xy} := \mathrm{vec}(w^{xy}_{i,j})$ and $\boldsymbol{w}_{yy} := \mathrm{vec}(w^{yy}_{i,j})$, five matrices $\boldsymbol{D}_x$, $\boldsymbol{D}_y$, $\boldsymbol{D}_{xx}$, $\boldsymbol{D}_{xy}$, and $\boldsymbol{D}_{yy}$ are the difference operators respectively satisfying

$$
\left.
\begin{aligned}
\boldsymbol{D}_x\boldsymbol{\Theta} &= \mathrm{vec}(\Theta_{i+1,j} - \Theta_{i,j}) \\
\boldsymbol{D}_y\boldsymbol{\Theta} &= \mathrm{vec}(\Theta_{i,j+1} - \Theta_{i,j}) \\
\boldsymbol{D}_{xx}\boldsymbol{\Theta} &= \mathrm{vec}(\Theta_{i+2,j} - 2\Theta_{i+1,j} + \Theta_{i,j}) \\
\boldsymbol{D}_{xy}\boldsymbol{\Theta} &= \mathrm{vec}(\Theta_{i+1,j+1} - \Theta_{i+1,j} - \Theta_{i,j+1} + \Theta_{i,j}) \\
\boldsymbol{D}_{yy}\boldsymbol{\Theta} &= \mathrm{vec}(\Theta_{i,j+2} - 2\Theta_{i,j+1} + \Theta_{i,j})
\end{aligned}
\right\},
$$

and $\boldsymbol{\delta}_x := \mathrm{vec}(W(\Theta^W_{i+1,j} - \Theta^W_{i,j}))$ and $\boldsymbol{\delta}_y := \mathrm{vec}(W(\Theta^W_{i,j+1} - \Theta^W_{i,j}))$ are the vectors of the unwrapped phase difference estimated from $(\Theta^W_{i,j})$. We obtain $\boldsymbol{\Theta}^*$ by the alternating direction method of multipliers (ADMM) [19] through the following ADMM-formulation:

$$
\boldsymbol{\Theta}^* \in \underset{\boldsymbol{\Theta}}{\mathrm{argmin}}\ \|\boldsymbol{D}_1\boldsymbol{\Theta} - \boldsymbol{\delta}\|_{1,\boldsymbol{w}_1} + \|\boldsymbol{D}_2\boldsymbol{\Theta}\|^2_{2,\boldsymbol{w}_2} + \varepsilon\,\|\boldsymbol{\Theta}\|_2^2,
$$

where $\boldsymbol{w}_1 := (\boldsymbol{w}_x^T, \boldsymbol{w}_y^T)^T$, $\boldsymbol{w}_2 := (\boldsymbol{w}_{xx}^T, \boldsymbol{w}_{xy}^T, \boldsymbol{w}_{yy}^T)^T$, $\boldsymbol{D}_1 := (\boldsymbol{D}_x^T, \boldsymbol{D}_y^T)^T$, $\boldsymbol{D}_2 := (\boldsymbol{D}_{xx}^T, \boldsymbol{D}_{xy}^T, \boldsymbol{D}_{yy}^T)^T$, $\boldsymbol{\delta} := (\boldsymbol{\delta}_x^T, \boldsymbol{\delta}_y^T)^T$, and $\varepsilon\,\|\boldsymbol{\Theta}\|_2^2$ $(0 < \varepsilon \ll 1)$ is added for regularization. The ADMM computes $\boldsymbol{\Theta}^*$ by the following iteration:

$$
\left\lfloor
\begin{aligned}
\boldsymbol{\Theta}_{k+1} &= \frac{1}{\gamma}\boldsymbol{K}^{-1}\boldsymbol{D}_1^T(\boldsymbol{\nu}_k - \boldsymbol{\xi}_k) \\
\boldsymbol{\nu}_{k+1} &= \mathrm{prox}_{\gamma\|\cdot - \boldsymbol{\delta}\|_{1,\boldsymbol{w}_1}}(\boldsymbol{D}_1\boldsymbol{\Theta}_{k+1} + \boldsymbol{\xi}_k) \\
\boldsymbol{\xi}_{k+1} &= \boldsymbol{\xi}_k + \boldsymbol{D}_1\boldsymbol{\Theta}_{k+1} - \boldsymbol{\nu}_{k+1}
\end{aligned}
\right.
$$

with $\gamma > 0$ and any initialization $\boldsymbol{\Theta}_0 \in \mathbb{R}^{(n+1)(m+1)}$, $\boldsymbol{\nu}_0 \in \mathbb{R}^{n(m+1)+(n+1)m}$ and $\boldsymbol{\xi}_0 \in \mathbb{R}^{n(m+1)+(n+1)m}$, where

$$
\boldsymbol{K} := \frac{1}{\gamma}\boldsymbol{D}_1^T\boldsymbol{D}_1 + 2\left(\boldsymbol{D}_2^T\boldsymbol{W}_2\boldsymbol{D}_2 + \varepsilon\boldsymbol{I}\right),
$$

$\boldsymbol{I}$ denotes the identity matrix, $\boldsymbol{W}_2$ is a diagonal matrix satisfying $[\boldsymbol{W}_2]_i = [\boldsymbol{w}_2]_i$, and $\mathrm{prox}_{\gamma\|\cdot-\boldsymbol{\delta}\|_{1,\boldsymbol{w}_1}} : \mathbb{R}^{n(m+1)+(n+1)m} \to \mathbb{R}^{n(m+1)+(n+1)m}$ is the proximity operator of $\gamma\,\|\cdot - \boldsymbol{\delta}\|_{1,\boldsymbol{w}_1}$ defined as

$$
\begin{aligned}
&[\mathrm{prox}_{\gamma\|\cdot-\boldsymbol{\delta}\|_{1,\boldsymbol{w}_1}}(\boldsymbol{\nu})]_i \\
&\quad := \begin{cases}
[\boldsymbol{\nu}]_i - \gamma[\boldsymbol{w}_1]_i & \text{if } [\boldsymbol{\nu}]_i \geq [\boldsymbol{\delta}]_i + \gamma[\boldsymbol{w}_1]_i, \\
[\boldsymbol{\nu}]_i + \gamma[\boldsymbol{w}_1]_i & \text{if } [\boldsymbol{\nu}]_i \leq [\boldsymbol{\delta}]_i - \gamma[\boldsymbol{w}_1]_i, \\
[\boldsymbol{\delta}]_i & \text{otherwise.}
\end{cases}
\end{aligned}
$$

3.2 Virtual Samples Generated in Step B

The minimizer $\boldsymbol{\Theta}^*$ obtained by the ADMM in Step A does not guarantee $W(\Theta^*_{i,j}) = \Theta^W_{i,j}$, and hence $\Delta\Theta_{i,j}$ tends to be smaller than the true unwrapped phase difference. Therefore we need to adjust $\Theta^*_{i,j}$ based on $\Theta^W_{i,j}$. The simplest adjustment is defining new unwrapped phase $\widehat{\Theta}_{i,j} := \widehat{\Theta}(x_i, y_j)$ as $\widehat{\Theta}_{i,j} := \Theta^*_{i,j} + W(\Theta^W_{i,j} - \Theta^*_{i,j})$, which satisfies

$$
\widehat{\Theta}_{i,j} = \underset{W(\Theta_{i,j}) = \Theta^W_{i,j}}{\mathrm{argmin}}\ |\Theta^*_{i,j} - \Theta_{i,j}|.
$$

However this method often destroys the smoothness of $\boldsymbol{\Theta}^*$, e.g., if $W(\Theta^W_{i,j} - \Theta^*_{i,j}) \approx \pi$ and $W(\Theta^W_{i+1,j} - \Theta^*_{i+1,j}) \approx -\pi$, then $\widehat{\Theta}_{i+1,j} - \widehat{\Theta}_{i,j} \approx \Theta^*_{i+1,j} - \Theta^*_{i,j} - 2\pi \not\approx \Theta^*_{i+1,j} - \Theta^*_{i,j}$.

In case of $W(\Theta_{0,0}^W - \Theta_{0,0}^*) \geq 0$, in the ideal situation for preserving the geometry of $\boldsymbol{\Theta}^*$, the following properties hold for all i and j.

- $W(\Theta_{i,j}^W - \Theta_{i,j}^*) \geq 0$.
- $W(\Theta_{i+1,j}^W - \Theta_{i+1,j}^*) \approx W(\Theta_{i,j}^W - \Theta_{i,j}^*)$.
- $W(\Theta_{i,j+1}^W - \Theta_{i+1,j+1}^*) \approx W(\Theta_{i,j}^W - \Theta_{i,j}^*)$.

Therefore if there exists (i, j) overly departs from the above situation, we decide that the wrapped sample $\Theta_{i,j}^W$ has large additive noise and define a new unwrapped phase sample $\widehat{\Theta}_{i,j} := \Theta_{i,j}^* + W(\Theta_{i,j}^W - \Theta_{i,j}^*) + \kappa$, where $\kappa \in (0, 2\pi]$. To wrap up, the new unwrapped phase samples $(\widehat{\Theta}_{i,j})$ is obtained by the following algorithm.

Algorithm 1: Adjustment of $\Theta_{i,j}^*$ based on $\Theta_{i,j}^W$

```
Input: (Θ*_{i,j}), (Θ^W_{i,j}), κ ∈ (0, 2π], and μ ∈ [0, 1]
Output: (Θ̂_{i,j})
α_{i,j} ← W(Θ^W_{i,j} − Θ*_{i,j}) for all i and j.
β_{i,j} ← W(Θ^W_{i,j} − Θ*_{i,j}) for all i and j.
for i = 1 to n do
    if α_{i,0} < 0 and |α_{i,0} + κ − α_{i−1,0}| < |α_{i,0} − α_{i−1,0}| then
        α_{i,0} ← α_{i,0} + κ.
    end if
end for
for j = 1 to m do
    if β_{0,j} < 0 and |β_{0,j} + κ − β_{0,j−1}| < |β_{0,j} − β_{0,j−1}| then
        β_{0,j} ← β_{0,j} + κ.
    end if
end for
α_{0,j} ← β_{0,j} for j = 1, ..., m.
β_{i,0} ← α_{i,0} for i = 1, ..., n.
for i = 1 to n
    for j = 1 to m
        if α_{i,j} < 0 and |α_{i,j} + κ − α_{i,j−1}| < |α_{i,j} − α_{i,j−1}| then
            α_{i,j} ← α_{i,j} + κ.
        end if
        if β_{i,j} < 0 and |β_{i,j} + κ − β_{i−1,j}| < |β_{i,j} − β_{i−1,j}| then
            β_{i,j} ← β_{i,j} + κ.
        end if
    end for
end for
Θ̂_{i,j} ← Θ*_{i,j} + μα_{i,j} + (1 − μ)β_{i,j} for all i and j.
```

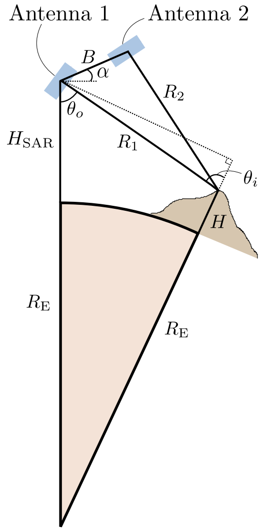
In case of $W(\Theta_{0,0}^W - \Theta_{0,0}) < 0$, $\widehat{\boldsymbol{\Theta}}$ is obtained in the same manner as Algorithm 1. Finally, we produce the virtual wrapped phase $\widehat{\Theta}_{i,j}^W := \widehat{\Theta}^W(x_i', y_j')$ at new regular rectangle grid points $\mathcal{G}' := \{(x_i', y_j') \mid i = 0, 1, \ldots, ln \text{ and } j = 0, 1, \ldots, lm\}$ s.t. $l \in \mathbb{Z}_+$, $l \geq 2$, $x_0' = x_0$, $x_{ln}' = x_n$, $x_{i+1}' - x_i' = h_x/l$ for all i, $y_0' = y_0$, $y_{ln}' = y_n$, and $y_{j+1}' - y_j' = h_y/l$ for all j, defined as

$$\widehat{\Theta}_{il+s,jl+t}^W := W\left(\widehat{\Theta}_{i,j} + s\,\frac{\widehat{\Theta}_{i+1,j} - \widehat{\Theta}_{i,j}}{l} + t\,\frac{\widehat{\Theta}_{i,j+1} + s\,\frac{\widehat{\Theta}_{i+1,j+1} - \widehat{\Theta}_{i,j+1}}{l} - \left(\widehat{\Theta}_{i,j} + s\,\frac{\widehat{\Theta}_{i+1,j} - \widehat{\Theta}_{i,j}}{l}\right)}{l}\right)$$

for $i = 0, 1, \ldots, n-1$, $j = 0, 1, \ldots, m-1$, $s = 0, 1, \ldots, l$, and $t = 0, 1, \ldots, l$. We apply the proposed phase unwrapping algorithm to $(\widehat{\Theta}_{i,j}^W)$ and construct θ_{f^*} as an estimate.

4 Terrain Height Estimation by InSAR System

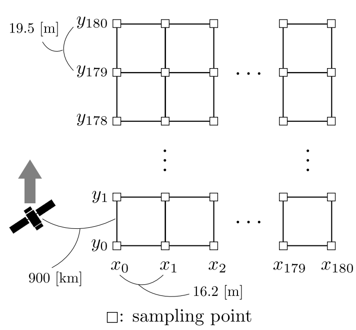
The interferometric synthetic aperture radar (InSAR) [3], [4] is an imaging technique allowing highly accurate measurements of a surface topography in all weather conditions, day or night. In the InSAR system, a pair of antennas, say



(a) Outline of InSAR

α	$\pi/6$	[rad]
λ	23.5	[cm]
B	500	[m]
R_{E}	6371	[km]
H_{SAR}	800	[km]

(b) Parameters of InSAR



(c) Setting of observation points

Figure 2: Outline and setting of the terrain height estimation

Antennas 1 and 2 (see Fig. 2(a)), on-board an aircraft or a spacecraft platform transmit coherent broadband microwave radio signals and receive the reflected signals from the same scene. Antennas 1 and 2 respectively receive

$$s_1 := |s_1| e^{-\imath\left(\frac{4\pi R_1}{\lambda} - \phi_1 + \nu_1\right)} \text{ and } s_2 := |s_2| e^{-\imath\left(\frac{4\pi R_2}{\lambda} - \phi_2 + \nu_2\right)},$$

where λ is the wavelength of the trasmitted signal, R_1 and R_2 are respectively the distance from Antennas 1 and 2 to the target, ϕ_1 and ϕ_2 are the backscatter phase delays, ν_1 and ν_2 are additive phase noise. Since the backscatter phase delays ϕ_1 and ϕ_2 are determined by the shape of the target, the geological condition, and the weather condition, if these conditions are same between two received signals, we have $\phi_1 = \phi_2$. Therefore we obtain interferometric image as

$$\bar{s}_1 s_2 = |s_1||s_2| e^{\imath\left(\frac{4\pi(R_1 - R_2)}{\lambda} + \nu\right)},$$

where $\bar{s}_1$ denotes the complex conjugate of s_1 and $\nu := \nu_1 - \nu_2$. From Fig. 2(a) and the law of cosines, the interferometric phase $\Theta_{\mathrm{int}} := \frac{4\pi(R_1 - R_2)}{\lambda}$ is expressed as

$$\Theta_{\mathrm{int}} = \frac{4\pi}{\lambda}\left\{R_1 - \sqrt{R_1^2 + B^2 - 2R_1 B \sin(\theta_o - \alpha)}\right\}.$$

Suppose that we know the height at (x_0, y_0) as H_0. Then we compute the reference phase defined as

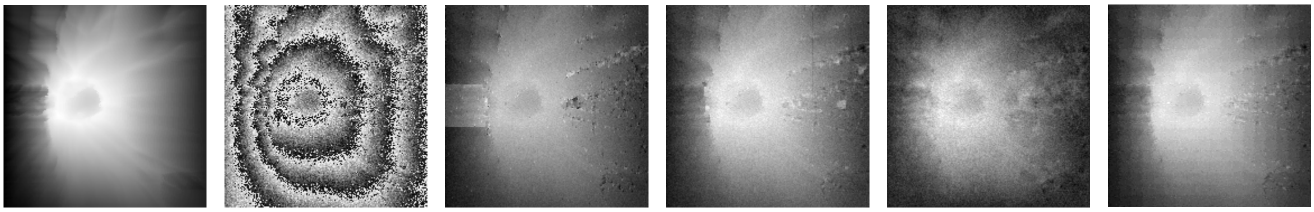
$$\Theta_{\mathrm{ref}} := \frac{4\pi}{\lambda}\left\{R_1 - \sqrt{R_1^2 + B^2 - 2R_1 B \sin(\theta_o^{H_0} - \alpha)}\right\}$$

s.t. $\theta_o^{H_0} := \arccos\left(\frac{R_1^2 + (R_{\mathrm{E}} + H_{\mathrm{SAR}})^2 - (R_{\mathrm{E}} + H_0)^2}{2R_1(R_{\mathrm{E}} + H_{\mathrm{SAR}})}\right)$, which is a virtual interferometric phase assuming that the terrain height is always H_0. Define an unknown 2D unwrapped phase as

$$\Theta := \Theta_{\mathrm{int}} - \Theta_{\mathrm{ref}} \approx \frac{4\pi B \cos(\theta_o^{H_0} - \alpha)}{\lambda R_1 \sin \theta_i^{H_0}}(H - H_0),$$

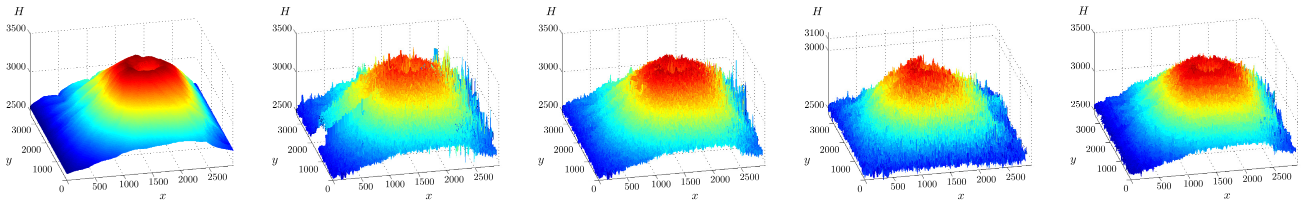
where $\theta_i^{H_0} := \arcsin\left(\frac{(R_{\mathrm{E}} + H_{\mathrm{SAR}}) \sin \theta_o^{H_0}}{R_{\mathrm{E}} + H_0}\right)$ and we can observe its noisy wrapped sample as $\Theta^W := W(\Theta_{\mathrm{int}} - \Theta_{\mathrm{ref}} + \nu) = W(\Theta_{\mathrm{int}}^W - \Theta_{\mathrm{ref}})$ [20], where $\Theta_{\mathrm{int}}^W := W(\Theta_{\mathrm{int}} + \nu)$.

Fig. 3(a) is the true unwrapped phase Θ generated from a virtual mountain shown in Fig. 4(a). The parameters



(a) Θ (b) Θ^W (c) BC (MSE = 1.4766) (d) MCF (MSE = 1.3168) (e) MLP (MSE = 50.6141) (f) Proposed (MSE = 0.6428)

Figure 3: Estimates $\widetilde{\Theta}$ of Θ from Θ^W and their mean square error $\mathrm{MSE} := \frac{1}{181\cdot 181}\sum_{i=0}^{180}\sum_{j=0}^{180}|\Theta(x_i,y_j) - \widetilde{\Theta}(x_i,y_j)|^2$



(a) Virtual mountain (b) BC (MAE = 57.5304) (c) MCF (MAE = 49.0457) (e) MLP (MAE = 256.0205) (e) Proposed (MAE = 34.7725)

Figure 4: Estimates $\widetilde{H}$ of the height H and their mean absolute error $\mathrm{MAE} := \frac{1}{181\cdot 181}\sum_{i=0}^{180}\sum_{j=0}^{180}|H(x_i,y_j) - \widetilde{H}(x_i,y_j)|$

of InSAR and the setting of $\mathcal{G}$ are respectively shown in Figs. 2(b) and 2(c). Figure 3(b) depicts the observed noisy wrapped samples, and Figs. 3(c), 3(d), 3(e), and 3(f) depict the estimates by the BC, the MCF, the MLP ($p = 2$), and the propose method ($\kappa = 3\pi/2$, $\mu = 1/2$, $l = 3$,$\epsilon_{i,j}^{(0)} = 1 - |\cos(\widehat{\Theta}_{i,j}^W)|$ and $\epsilon_{i,j}^{(1)} = 1 - |\sin(\widehat{\Theta}_{i,j}^W)|$) respectively. Figures 4(b), 4(c), 4(d), and 4(e) show the estimates of the terrain height based on the estimated unwrapped phases. From Figs. 3 and 4, we observe that the proposed phase unwrapping algorithm gives the best performance compared the other algorithms visually and numerically.

5 Conclusion

In this paper, we have proposed a preprocessing of the algebraic 2D phase unwrapping algorithm which needs to construct a smooth spline function not having zeros. The proposed preprocessing was implemented by finding a minimizer of a convex cost function and producing virtual wrapped phase based on the minimizer and the observed wrapped phase. The simulation of the terrain height estimation showed the effectiveness of the proposed method.

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