Algebraic Phase Unwrapping over Collection of Triangles and Its Application

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Abstract Phase unwrapping is an estimation problem of the continuous phase function from its wrapped samples. Especially the two-dimensional phase unwrapping has been a common key for estimating many crucial physical information, e.g., the surface topography measured by interferometric synthetic aperture radar/sonar. However almost all 2D phase unwrapping algorithms are suffering from the path dependence of the estimated result mainly because they tackle a certain NP-hard problem. In this paper, to guarantee the path independence, we present a novel algebraic approach by combining the ideas in the algebraic phase unwrapping with techniques for a piecewise polynomial interpolation of two-dimensional finite data sequence.

1 Introduction

Two-dimensional (2D) phase unwrapping [1] is an estimation problem of the unknown unwrapped phase $\Theta(x, y) \in \mathbb{R}$ defined in a simply connected region $\Omega \subset \mathbb{R}^2$, from its finite wrapped samples $[\Theta(x, y)]_{\mathrm{mod}\ 2\pi} \in (-\pi, \pi]$ observed at $(x, y) \in \mathcal{G}(\subset \Omega)$, where $\mathcal{G}$ stands for the set of finite grid points. The 2D phase unwrapping has been a common key for estimating many crucial physical information such as the surface topography measured by interferometric synthetic aperture radar (InSAR) [2]–[4] or interferometric synthetic aperture sonar (InSAS) [5], the degree of magnetic field in homogeneity in the water/fat separation problem in magnetic resonance imaging (MRI) [6], and the accurate profile of mechanical parts by x-ray [7].

Almost all existing 2D phase unwrapping algorithms [1] estimate the unwrapped phase as $\widetilde{\Theta}(x, y) := [\Theta(x, y)]_{\mathrm{mod}\ 2\pi} + 2\pi\eta(x, y)$ with $\eta : \mathcal{G} \to \mathbb{Z}$, by trying to find η^* which minimizes the cardinality of

$$\Big\{(x, y) \mid \big|\widetilde{\Theta}(x, y) - \widetilde{\Theta}(x', y')\big| > \pi, \quad \text{where } (x', y') \text{ is a neighboring grid point of } (x, y)\Big\}.$$

Unfortunately, this combinatorial problem is intractable due to its NP-Hardness [4]. As a result, such algorithms are suffering from the so-called *path dependence* of the estimated unwrapped phase, i.e., the estimated result differs depending on the execution procedure of the algorithm. This situation implies that any technically reliable solution has not yet been established even though the failure of 2D phase unwrapping makes a substantial impact on the accuracy of the estimated physical information.

In this paper, we propose a completely different algebraic approach to the 2D phase unwrapping problem. We estimate Θ by θ_f, where θ_f is the unwrapped phase of a twice differentiable complex function $f := f_{(0)} + j f_{(1)}$. Then the estimation problem of Θ is replaced with that of f. The proposed approach is designed based on Theorem 1 in Sect. 3.2 which was established recently in [8] to guarantee the unique existence of $\theta_f \in C^2(\Omega)$ as a scalar potential having, as its gradient flow, the partial derivatives of the wrapped phase function. In the spirit of functional data analysis: *"smoothness of estimate should be measured for functions which possibly generate the data"* [9], we use best smoothing functions $f^*_{(0)}$ and $f^*_{(1)}$ among all possible candidates, in a suitable functional space, which are consistent with given wrapped phase information. As a result, we obtain a best scalar potential θ^*_f as an estimate of the unwrapped phase surface.

The proposed algorithm is realized by combining the ideas in the algebraic phase unwrapping [8], [10]–[12] with techniques for a piecewise polynomial interpolation [13]–[16] of two-dimensional finite data sequence, as a best solution of a variational problem. Remarkably, unlike almost all existing algorithms, the proposed algorithm guarantees the path independence of the estimated unwrapped phase under reasonable assumption.

A numerical experiment, based on the InSAR simulation, demonstrates the effectiveness of the proposed 2D phase unwrapping.

2 Preliminaries

2.1 Notation and the multivariate spline space

Let $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote respectively the set of all integers, real numbers, and complex numbers. We use $j \in \mathbb{C}$ to denote the imaginary unit satisfying $j^2 = -1$. For any $c \in \mathbb{C}$, $\Im(c)$ stands for the imaginary part of c.

Let Δ be a collection of triangles in whose union forms a simply connected region $\Omega \subset \mathbb{R}^2$. For any two triangles $\mathcal{T}, \mathcal{T}' \in \Delta$, if $\mathcal{T} \cap \mathcal{T}'$ is either empty or a common edge of $\mathcal{T}$ and $\mathcal{T}'$ or a common vertex of $\mathcal{T}$ and $\mathcal{T}'$, Δ is called a *regular triangulation*.

Given two integers $d \geq 0$ and $0 \leq \rho < d$, define

$$\mathcal{S}^\rho_d(\Delta) := \{f \in C^\rho(\Omega) \mid \text{for all } \mathcal{T} \in \Delta,\ f = f_{\mathcal{T}} \in \mathbb{P}_d \text{ over } \mathcal{T}\}$$

as the multivariate spline space of degree d and smoothness ρ, where $\mathbb{P}_d$ denotes the space of all polynomials whose degree is d at most.

2.2 The B-form representation of spline functions

Let $\mathcal{T} = \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle$ be a triangle, i.e., $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are not arranged linearly, in Δ with $\boldsymbol{v}_k = (x_k, y_k)$ $(k = 1, 2, 3)$. It is well-known that every point (x, y) can be expressed uniquely in the form

$$(x, y) = r\boldsymbol{v}_1 + s\boldsymbol{v}_2 + t\boldsymbol{v}_3 \qquad \text{s.t. } r + s + t = 1,$$

where (r, s, t) are called the *barycentric coordinates* of the point (x, y) with respect to the triangle $\mathcal{T}$.

For integers $l \geq 0$, $m \geq 0$ and $n \geq 0$, the *Bernstein-Bézier polynomials* are defined as

$$B^d_{l,m,n}(r, s, t) := \frac{d!}{l!m!n!} r^l s^m t^n \qquad \text{s.t. } l + m + n = d.$$

The set of Bernstein-Bézier polynomials $B^d_{l,m,n}$ is a basis of the space $\mathbb{P}_d$. As a result, any spline function $f \in \mathcal{S}^\rho_d(\Delta)$ restricted to each triangle $\mathcal{T} \in \Delta$ can be written uniquely as

$$f_{\mathcal{T}}(r, s, t) = \sum_{l+m+n=d} c^{\mathcal{T}}_{l,m,n} B^d_{l,m,n}(r, s, t),$$

where $c^{\mathcal{T}}_{l,m,n} \in \mathbb{R}$. Such a representation is called the *B-form representation* of the spline function f. We denote the B-coefficient vector of f by $\boldsymbol{c} := \{c^{\mathcal{T}}_{l,m,n} \mid l + m + n = d,\ \mathcal{T} \in \Delta\}$.

Example 1 *Let us consider a simple example, where the triangulation Δ has only one triangle $\mathcal{T} = \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle = \Omega$, i.e., $\Delta := \{\mathcal{T}\}$. Suppose that the degree of the spline space is 4, and the B-coefficient vector $\boldsymbol{c} := (c_1, c_2, \ldots, c_{15})^T$ is defined as Fig. 1. Then spline function $f = f_{\mathcal{T}}$ is expressed, by use of the barycentric coordinates (r, s, t) with respect to the triangle $\mathcal{T}$, as Eq. (1).*

3 2D Phase Unwrapping as a Scalar Potential

3.1 Setting of the set $\mathcal{G}$ and the triangulation Δ

In what follows, assume that the set of finite sampling points is a regular rectangular grid on the area $\Omega := [a, b] \times [c, d]$, i.e., $\mathcal{G} = \{(x_{k_1}, y_{k_2})\}_{k_1,k_2}$ to be given by $a =: x_0 < x_1 < \ldots < x_p := b$ and $c =: y_0 < y_1 < \ldots < y_q := d$ satisfying $x_{k_1+1} - x_{k_1} = y_{k_2+1} - y_{k_2} = 1$ ($k_1 = 0, 1, \ldots, p - 1$ and $k_2 = 0, 1, \ldots, q - 1$). Moreover, we construct a triangulation $\Delta_\dagger$ by dividing every regular rectangular $[x_{k_1}, y_{k_2}] \times [x_{k_1+1}, y_{k_2+1}]$ into four triangles as Fig. 2. This triangulation is called the *crisscross partition*.

According to [15], the minimal degree of the spline space $\mathcal{S}^2_d(\Delta_\dagger)$ is $d = 4$ for twice differentiability and interpolating a given data on $\mathcal{G}$. For $\mathcal{S}^2_4(\Delta_\dagger)$, in every regular regular rectangular $[x_{k_1}, y_{k_2}] \times [x_{k_1+1}, y_{k_2+1}]$, there are 25 B-coefficients within the rectangular, 4 B-coefficients at sampling points, and the other 6 B-coefficients on sides of rectangular as shown in Fig. 2. Hence the number of B-coefficients is

$$\begin{aligned} n_c &= 25pq + (p + 1)(q + 1) + 3p(q + 1) + 3(p + 1)q \\ &= 32pq + 4p + 4q + 1, \end{aligned}$$

i.e., $\boldsymbol{c} \in \mathbb{R}^{n_c}$.

3.2 Strategy of the proposed method

The next theorem, derived by using Poincaré's lemma [17], motivated us to formulate of 2D phase unwrapping as Problem 1 below.

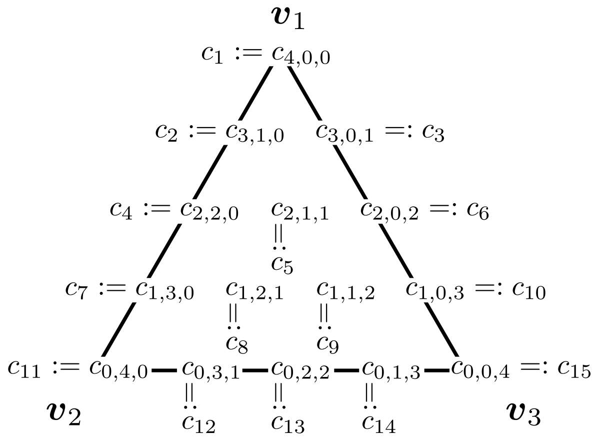


Figure 1: A typical triangle $\mathcal{T} = \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle$ with associated coefficients for degree 4

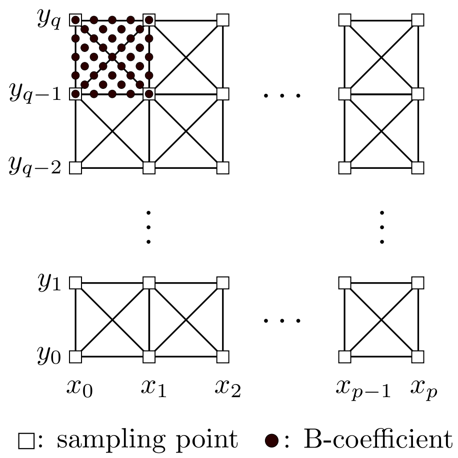


Figure 2: Crisscross partition $\Delta_\dagger$ for the rectangular area Ω

Theorem 1 (Two-dimensional phase unwrapping as a scalar potential function [8]) *Suppose that $f_{(i)} : \mathbb{R}^2 \to \mathbb{R}$ $(i = 0, 1)$ are $C^2(\Omega)$ functions satisfying $f(x, y) := f_{(0)}(x, y) + jf_{(1)}(x, y) \neq 0$ for all $(x, y) \in \Omega$. Then for an arbitrarily fixed $(x_0, y_0) \in \Omega$ and $\theta_0 \in (-\pi, \pi]$ satisfying $f(x_0, y_0) = |f(x_0, y_0)|e^{j\theta_0}$, the following hold.*

(a) *There exists a unique $\theta_f \in C^2(\Omega)$ satisfying $\theta_f(x_0, y_0) = \theta_0$ and for all $(x, y) \in \Omega$*

$$\left.\begin{aligned} \frac{\partial \theta_f}{\partial x}(x, y) &= \Im\left[\frac{\frac{\partial f_{(0)}}{\partial x}(x, y) + j\frac{\partial f_{(1)}}{\partial x}(x, y)}{f_{(0)}(x, y) + jf_{(1)}(x, y)}\right] \\ \frac{\partial \theta_f}{\partial y}(x, y) &= \Im\left[\frac{\frac{\partial f_{(0)}}{\partial y}(x, y) + j\frac{\partial f_{(1)}}{\partial y}(x, y)}{f_{(0)}(x, y) + jf_{(1)}(x, y)}\right] \end{aligned}\right\}.$$

In other words, θ_f is a scalar potential of

$$\left(\Im\left[\frac{\frac{\partial f_{(0)}}{\partial x}(x, y) + j\frac{\partial f_{(1)}}{\partial x}(x, y)}{f_{(0)}(x, y) + jf_{(1)}(x, y)}\right], \Im\left[\frac{\frac{\partial f_{(0)}}{\partial y}(x, y) + j\frac{\partial f_{(1)}}{\partial y}(x, y)}{f_{(0)}(x, y) + jf_{(1)}(x, y)}\right]\right).$$

(b) *$\theta_f \in C^2(\Omega)$, defined in* (a), *is given by*

$$\theta_f(x, y) = \theta_f(x_0, y_0) + \int_0^1 \Im\left[\frac{(f_{(0)}(\gamma(t)))' + j\,(f_{(1)}(\gamma(t)))'}{f_{(0)}(\gamma(t)) + jf_{(1)}(\gamma(t))}\right] dt, \quad (2)$$

$$f_{\mathcal{T}}(r,s,t) = c_1 r^4 + 4c_2 r^3 s + 4c_3 r^3 t + 6c_4 r^2 s^2 + 12c_5 r^2 st + 6c_6 r^2 t^2 + 4c_7 rs^3 + 12c_8 rs^2 t + 12c_9 rst^2 + 4c_{10} rt^3 + c_{11} s^4 + 4c_{12} s^3 t + 6c_{13} s^2 t^2 + 4c_{14} st^3 + c_{15} t^4 \quad (1)$$

where $\gamma : [0,1] \to \Omega$ is any piecewise C^1 path satisfying $\gamma(0) = (x_0, y_0)$ and $\gamma(1) = (x, y)$.

From Theorem 1, we can reduce the estimation problem of Θ to that of $f_{(i)} \in C^2(\Omega)$ $(i = 0, 1)$ satisfying $f(x,y) = f_{(0)}(x,y) + jf_{(1)}(x,y) \neq 0$ for all $(x,y) \in \Omega$. Hence we can design θ_f, as an ideal estimate of Θ in the sense of the functional data analysis: *"smoothness of estimate should be measured for functions which possibly generate the data"* [9], by smoothing $f_{(0)} : (x,y) \mapsto a(x,y)\cos(\theta_f(x,y))$ and $f_{(1)} : (x,y) \mapsto a(x,y)\sin(\theta_f(x,y))$ in $\mathcal{S}_4^2(\Delta_\dagger)$, subject to $a(x,y) > 0$ $(\forall (x,y) \in \Omega)$ and

$$\left.\begin{array}{l} f_{(0)}(x,y) = \cos([\Theta(x,y)]_{\text{mod } 2\pi}) \\ f_{(1)}(x,y) = \sin([\Theta(x,y)]_{\text{mod } 2\pi}) \end{array}\right\} \text{ for all } (x,y) \in \mathcal{G}.$$

As a result, we consider 2D phase unwrapping as the following problem which consists of two steps.

Problem 1

Step 1 *Find $f^*_{(i)} \in \mathcal{S}_4^2(\Delta_\dagger)$ $(i = 0, 1)$ which minimizes*

$$J(f_{(i)}) := \iint_\Omega \left[\left| \frac{\partial^2 f_{(i)}}{\partial x^2} \right|^2 + 2 \left| \frac{\partial^2 f_{(i)}}{\partial x \partial y} \right|^2 + \left| \frac{\partial^2 f_{(i)}}{\partial y^2} \right|^2 \right] dxdy \quad (3)$$

subject to

$$\left.\begin{array}{l} f_{(0)}(x,y) = \cos([\Theta(x,y)]_{\text{mod } 2\pi}) \\ f_{(1)}(x,y) = \sin([\Theta(x,y)]_{\text{mod } 2\pi}) \end{array}\right\} \text{ for all } (x,y) \in \mathcal{G}.$$

Step 2 *For any point of interest $(x,y) \in \Omega$, compute $\theta_{f^*}(x,y)$ defined by (2) along a suitable piecewise C^1 path γ.*

Remark 1 *Problem 1 is a convex relaxation of an ideal optimization problem which includes an additional constraint $f_{(0)} + jf_{(1)} \neq 0$ over Ω. Fortunately, if sufficiently many grid points are employed for Θ, the solution $(f^*_{(0)}, f^*_{(1)})$ of this relaxed problem tends to satisfy automatically $f^*_{(0)} + jf^*_{(1)} \neq 0$ over Ω because the sum of squares achieves 1 at every grid points and the rapid local change $J(f^*_{(i)})$ $(i = 0, 1)$ are suppressed globally.*

3.3 Solution of Step 1 in Problem 1

As shown in [18], the energy expression (3) can be written as quadratic form $J(f_{(i)}) = J(\boldsymbol{c}_{(i)}) = \boldsymbol{c}_{(i)}^T \boldsymbol{Q} \boldsymbol{c}_{(i)}$, where $\boldsymbol{Q} \in \mathbb{R}^{n_c \times n_c}$ is a symmetric positive semi-definite matrix. Moreover, the condition $f_{(i)} \in \mathcal{S}_4^2(\Delta_\dagger)$ and the interpolating condition can be respectively written as $\boldsymbol{\mathcal{H}} \boldsymbol{c}_{(i)} = \boldsymbol{0}$ and $\boldsymbol{\mathcal{I}} \boldsymbol{c}_{(i)} = \boldsymbol{d}_{(i)}$, where $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{I}}$ are certain sparse matrices [14]–[16], and $\boldsymbol{d}_{(i)} \in \mathbb{R}^{(p+1)(q+1)}$ $(i = 0, 1)$ is given by

$$\left.\begin{array}{l} \boldsymbol{d}_{(0)} = \left\{ \cos([\Theta(x_{k_1}, y_{k_2})]_{\text{mod } 2\pi}) \right\}_{k_1, k_2} \\ \boldsymbol{d}_{(1)} = \left\{ \sin([\Theta(x_{k_1}, y_{k_2})]_{\text{mod } 2\pi}) \right\}_{k_1, k_2} \end{array}\right\}.$$

Therefore, Step 1 in Problem 1 is replaced with the following convex optimization problem.

Step 1 *Find $\boldsymbol{c}^*_{(i)} \in \mathbb{R}^{n_c}$ $(i = 0, 1)$ which minimizes*

$$\boldsymbol{c}_{(i)}^T \boldsymbol{Q} \boldsymbol{c}_{(i)} \qquad \textit{s.t. } \boldsymbol{\mathcal{H}} \boldsymbol{c}_{(i)} = \boldsymbol{0} \textit{ and } \boldsymbol{\mathcal{I}} \boldsymbol{c}_{(i)} = \boldsymbol{d}_{(i)}.$$

To solve the above problem, we use the following iteration method introduced in [19]

$$\boldsymbol{c}_{(i)}^{(1)} = \frac{1}{\epsilon} \left(\boldsymbol{Q} + \frac{1}{\epsilon} \left(\boldsymbol{\mathcal{H}}^T \boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{I}}^T \boldsymbol{\mathcal{I}} \right) \right)^{-1} \boldsymbol{\mathcal{I}}^T \boldsymbol{d}_{(i)}$$

$$\boldsymbol{c}_{(i)}^{(k+1)} = \left(\boldsymbol{Q} + \frac{1}{\epsilon} \left(\boldsymbol{\mathcal{H}}^T \boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{I}}^T \boldsymbol{\mathcal{I}} \right) \right)^{-1} \left(\boldsymbol{Q} \boldsymbol{c}_{(i)}^{(k)} + \frac{1}{\epsilon} \boldsymbol{\mathcal{I}}^T \boldsymbol{d}_{(i)} \right)$$

for $k \geq 1$ and $\epsilon > 0$.

3.4 Solution of Step 2 in Problem 1

By the path independence guaranteed by Theorem 1 and the choice of bivariate splines of degree 4 for $(f^*_{(0)}, f^*_{(1)})$, the line integral (2), can be decomposed into a finite sum of integrals of the following type:

$$\int_a^{t^*} \Im \left[\frac{A'_{(0)}(t) + jA'_{(1)}(t)}{A_{(0)}(t) + jA_{(1)}(t)} \right] dt,$$

where $A_{(0)}$ and $A_{(1)}$ are univariate real polynomials satisfying $A_{(0)}(t) + jA_{(1)}(t) \neq 0$ $(\forall t \in [a, t^*])$. Fortunately, a closed form expression of the integral, for nontrivial cases: $A_{(0)} \not\equiv 0$ and $A_{(1)} \not\equiv 0$, is given by the next theorem.

Theorem 2 (Algebraic phase unwrapping [8], [10]–[12])

$$\int_a^{t^*} \Im \left[\frac{A'_{(0)}(t) + jA'_{(1)}(t)}{A_{(0)}(t) + jA_{(1)}(t)} \right] dt$$

$$= \left\{ \begin{array}{ll} \arctan\left\{ \frac{A_{(1)}(t^*)}{A_{(0)}(t^*)} \right\} + [V\{\Psi(t^*)\} - V\{\Psi(a)\}]\pi & \textit{if } A_{(0)}(t^*) \neq 0 \\ \pi/2 + [V\{\Psi(t^*)\} - V\{\Psi(a)\}]\pi & \textit{if } A_{(0)}(t^*) = 0 \end{array} \right\}$$

$$- \left\{ \begin{array}{ll} \arctan\left\{ \frac{A_{(1)}(a)}{A_{(0)}(a)} \right\} & \textit{if } A_{(0)}(a) \neq 0 \\ \mathrm{sgn}(\Psi_0(a)\Psi_1(a))\pi/2 & \textit{if } A_{(0)}(a) = 0 \end{array} \right\},$$

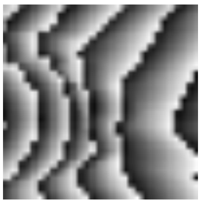
where $\mathrm{sgn}(t) = t/|t|$ for $t \neq 0$, $\mathrm{sgn}(t) = 0$ for $t = 0$, and $V\{\Psi(t)\}$ is the number of sign changes in polynomials $\{\Psi_0(t), \Psi_1(t), \ldots, \Psi_q(t)\}$ generated by Algorithm 1 below.

Algorithm 1 Sturm generating algorithm along the real axis (Sturm-$\mathcal{R}$)

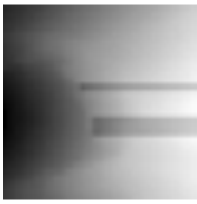
Input: $A_{(0)}(t), A_{(1)}(t) \in \mathbb{R}[t]$ and $a \in \mathbb{R}$ under the assumption in Sect. 3.4

1: $\Psi_0(t) \leftarrow \dfrac{A_{(0)}(t)}{(t-a)^{e_0}}$, $\Psi_1(t) \leftarrow \dfrac{A_{(1)}(t)}{(t-a)^{e_1}}$
   (e_i : the order of $t = a$ as a zero of polynomial $A_{(i)}(t)$ $(i = 0, 1)$)
2: $k \leftarrow 1$
3: **while** $\deg(\Psi_k) \neq 0$ **do**
4: $\quad \Psi_{k+1}(t) \leftarrow -\Psi_{k-1}(t) - H_k(t)\Psi_k(t)$
   (where $H_k(t) \in \mathbb{R}[t]$ and $\deg(\Psi_{k+1}) < \deg(\Psi_k)$)
5: $\quad k \leftarrow k + 1$
6: **end while**
7: $q \leftarrow \left\{ \begin{array}{ll} k & \text{if } \Psi_k(t) \not\equiv 0 \\ k-1 & \text{if } \Psi_k(t) \equiv 0 \end{array} \right.$

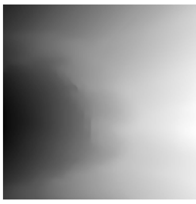
Output: $\{\Psi_k(t)\}_{k=0}^q$



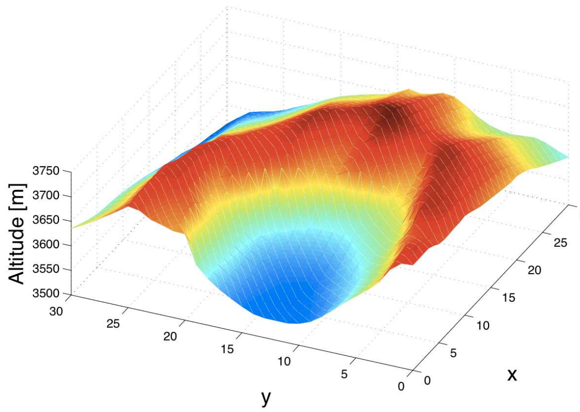
(a) Wrapped samples



(b) Estimation by $\widetilde{\Theta}$



(c) Estimation by θ_{f^*}



(d) Surface topography obtained by proposed algorithm

Figure 3: Comparison of proposed algorithm with conventional algorithm in InSAR

Remark 2 *We can compute the integral (2) for* $(f^*_{(0)}, f^*_{(1)})$ *without requiring any knowledge on the location of zeros of* $f^*_{(0)}$. *Proposition 2 was derived by extending Sturm's genius use [20] of the Euclidean algorithm for extraction of the root location of a real polynomial.*

4 Numerical Experiment

In this section, by the simulation of surface topographical mapping by InSAR system, we show the effectiveness of proposed 2D phase unwrapping algorithm. Fig. 3(a) depicts the wrapped samples observed at (x_{k_1}, y_{k_2}) ($k_1 = 0, 1, \ldots, 30$ and $k_2 = 0, 1, \ldots, 30$). Figs. 3(b) and (c) respectively depict the estimations of Θ by conventional algorithm and by proposed algorithm. From Figs. 3(b) and (c), we observe that the conventional algorithm failed in the recovery of smooth phase surface while the proposed algorithm obtained the smooth phase surface θ_{f^*}. Moreover, by doing the appropriate post-processing for θ_{f^*}, we can obtain the complete surface topography of the volcano in Shizuoka as shown in Fig. 3(d).

5 Conclusion

In this paper, we have proposed a novel algebraic approach to the 2D phase unwrapping. Remarkably, unlike almost all existing algorithms, the proposed approach guarantees the path independence of the estimated unwrapped phase under certain conditions. The effectiveness of the proposed approach have been confirmed by numerical example.

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