A Robust Algebraic Phase Unwrapping Based on Spline Approximation

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Abstract The phase unwrapping is a problem to find, at any specified point, the value of the continuous phase function which contains valuable information in many applications. If a given data sequence of 2-D real vectors is modeled as samples of a complex polynomial, the exact unwrapped phase can be computed through algorithms named the algebraic phase unwrapping. In this paper, to promote the understanding and extend the practicability of the algebraic phase unwrapping, we propose a widely applicable phase unwrapping technique, by combining the spline smoothing and the algebraic phase unwrapping, for a given data sequence of 2-D noisy vectors. The spline smoothing works as an optimal preprocessing in the sense that it is the unique solution to a variational problem for minimizing the sum of "fidelity" to the data and "roughness" of the curve. Fortunately, since the standard spline smoothing and its various generalizations produce always low-order piecewise real polynomials, we can compute the exact unwrapped phase for the pair of piecewise polynomials without suffering from a certain numerical instability observed typically in applications of the algebraic phase unwrapping to a complex polynomial of large degree.

Keywords Algebraic phase unwrapping, Spline smoothing, Piecewise polynomial, Cubic spline.

1. Introduction

For a given pair of continuous functions $f_{(i)} : \mathbb{R}^n \to \mathbb{R}$ $(i = 0, 1)$ and a certain continuous function (path of integration) $\Upsilon : \mathbb{R} \to \mathbb{R}^n$, such that

$$\left.\begin{array}{l} F_{(0)}(t) := f_{(0)}(\Upsilon(t)) \neq 0 \text{ or } F_{(1)}(t) := f_{(1)}(\Upsilon(t)) \neq 0 \\ \qquad\qquad\qquad\qquad \text{for all } t \in [t_0, t_1] \\ F_{(0)}(t_0) = f_{(0)}(\Upsilon(t_0)) \neq 0 \end{array}\right\}, \quad (1)$$

the unwrapped phase of $(f_{(0)}, f_{(1)})$ at $t^* \in (t_0, t_1]$ is defined as

$$\theta_F(t^*) := \theta_F(t_0) + \int_{t_0}^{t^*} \left(\arctan\left\{ \frac{F_{(1)}(t)}{F_{(0)}(t)} \right\} \right)' dt, \quad (2)$$

where $\theta_F(t_0) \in (-\pi, \pi]$ satisfies $F_{(0)}(t_0) + jF_{(1)}(t_0) = |F_{(0)}(t_0) + jF_{(1)}(t_0)| e^{j\theta_F(t_0)}$.

In many signal and image processing problems, the phase unwrapping has been a key for estimating some physical quantity, for example, surface topography in synthetic aperture radar (SAR) interferometry [1], [2], wavefront distortion in adaptive optics [3], [4], the degree of magnetic field inhomogeneity in the water/fat separation problem of magnetic resonance imaging (MRI) [5], [6] and the relationship between the object phase and the bispectrum phase in astronomical imaging [7], [8].

Despite the tremendous effort made so far, most phase unwrapping algorithms, e.g., path-following methods [9], [10] or minimum-norm methods [11], [12], are not necessarily designed based on sound mathematical analysis.

On the other hand, if the function $f_{(0)} + jf_{(1)} : \mathbb{R}^2 \to \mathbb{C}$ is given as a univariate complex polynomial, i.e., $f_{(0)}(x, y) + jf_{(1)}(x, y) \in \mathbb{C}[x + jy]$, together with its path of integration $\Upsilon(t) = (t, 0)$, $\Upsilon(t) = (0, t)$ or $\Upsilon(t) = (\cos t, \sin t)$, the algebraic phase unwrapping computes $\theta_F(t^*)$ exactly [13]-[15]. The algorithms do not require any numerical root finding or numerical integration technique.(1) Unfortunately, due to the poor performance of the complex polynomial as a nonparametric regression of a given data sequence of 2-D real vectors, the application of the algebraic phase unwrapping has been limited.

In this paper, we first present an extension of the algebraic phase unwrapping to a pair of piecewise real polynomials and then propose to apply this extension to the so-called spline smoothing of a given data sequence of 2-D real vectors. The spline smoothing works as an optimal preprocessing of the extended algebraic phase unwrapping because this smoothing is characterized as the unique solution to a variational problem for minimizing the sum of "fidelity" to the data and "roughness" of the curve (see e.g. [17]-[21] for such characterization and [22]-[24] for its applications of the spline smoothing to noise removal problems). Fortunately, since the standard spline smoothing and its various generalizations produce always low-order piecewise real polynomials, we can naturally apply the extended algebraic phase unwrapping to the pair of such piecewise real polynomials.

As a result, the proposed method consists of two steps. First, we compute a pair of spline functions which is expected to play

(1): Moreover, recent studies [16] reduce significantly a certain numerical instability, of the algebraic phase unwrapping, observed in its application to complex polynomials of large degree.

as a reliable nonparametric regression for a given data sequence of 2-D real vectors. Second, we apply the extended algebraic phase unwrapping to the pair of spline functions for computing the unwrapped phase.

Numerical examples demonstrate the practical applicability of the proposed technique even for noisy data samples.

2. Preliminaries

2.1 Algebraic phase unwrapping

Let $\mathbb{N}^*$, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the set of all positive integers, real numbers and complex numbers. We use $j \in \mathbb{C}$ to denote the imaginary unit satisfying $j^2 = -1$.

The next proposition presents the exact solution of the phase unwrapping problem for complex polynomials along the real axis.

[Proposition 1] (Algebraic phase unwrapping along the real axis [15]) *Let $A(t) := A_{(0)}(t) + jA_{(1)}(t) \in \mathbb{C}[t]$ satisfy $A(t) \neq 0$ $(t \in [t_a, t_b])$ and $A_{(0)}(t_a), A_{(0)}(t_b) \neq 0$ for some $t_b > t_a$, where $A_{(0)}(t), A_{(1)}(t) \in \mathbb{R}[t]$. Define*

$$\mathcal{Z}^{\dagger}_{A_{(0)}} := \{t \in [t_a, t_b] \mid A_{(0)}(t) = 0\} = \begin{cases} \emptyset & \text{if } A_{(0)}(t) \neq 0 \text{ for all } t \in [t_a, t_b], \\ \{\nu_1, \nu_2, \ldots, \nu_\tau\} & \text{otherwise}, \end{cases}$$

where $\nu_0 (:=) t_a < \nu_1 < \cdots < \nu_\tau < t_b$, and

$$\mathcal{X}(\nu_i) := \begin{cases} +1 & \text{if } \begin{cases} A_{(0)}(t)A_{(1)}(t) > 0 & \text{for } t \in (\nu_i - \varepsilon, \nu_i) \text{ and} \\ A_{(0)}(t)A_{(1)}(t) < 0 & \text{for } t \in (\nu_i, \nu_i + \varepsilon), \end{cases} \\ -1 & \text{if } \begin{cases} A_{(0)}(t)A_{(1)}(t) < 0 & \text{for } t \in (\nu_i - \varepsilon, \nu_i) \text{ and} \\ A_{(0)}(t)A_{(1)}(t) > 0 & \text{for } t \in (\nu_i, \nu_i + \varepsilon), \end{cases} \\ 0 & \text{otherwise}, \end{cases}$$

for ν_i $(i = 1, 2, \ldots, \tau)$ and for sufficiently small $\varepsilon > 0$. Then we have the following relations.

(a) *For any $t^* \in (t_a, t_b]$,*

$$\begin{aligned} \theta_A(t^*) &= \theta_A(t_a) + \int_{t_a}^{t^*} \left(\arctan \left\{ \frac{A_{(1)}(t)}{A_{(0)}(t)} \right\} \right)' dt \qquad (3) \\ &= \theta_A(t_a) - \arctan\{\mathcal{Q}_A(t_a)\} \\ &\quad + \lim_{t \to t^* - 0} \arctan\{\mathcal{Q}_A(t)\} + \Lambda(t^*)\pi, \end{aligned}$$

where $\mathcal{Q}_A(t) := \frac{A_{(1)}(t)}{A_{(0)}(t)}$ and $\Lambda(t^*) := \sum_{\nu_i \in (t_a, t^*)} \mathcal{X}(\nu_i)$.

(b) *Let $\{\Psi_k\}_{k=0}^q$ be a sequence of functions obtained by applying the algorithm (SGA) in Fig. 1 to $A_{(0)}(t)$ and $A_{(1)}(t)$ with*

$$\Psi_k(t) := \frac{\widehat{\Psi}_k(t)}{(t - t_a)^{e_k}},$$

where e_k denotes the order of $t = t_a$ as a zero of polynomial $\widehat{\Psi}_k(t)$. Define for each $t \in [t_a, t_b]$ the number of variations in the sign of $\{\Psi_k(t)\}_{k=0}^q$ by

$$\begin{aligned} V\{\Psi(t)\} &:= V\{\Psi_0(t), \Psi_1(t), \ldots, \Psi_q(t)\} \\ &:= \left| \{ i \mid 0 \le i < q \text{ and } \Psi_i(t)\Psi_{i+\varrho(i)}(t) < 0 \} \right|, \end{aligned}$$

begin
  Let $\Psi_0(t) := A_{(0)}(t)$ **and** $\widehat{\Psi}_1(t) := A_{(1)}(t)$,
  where $A_{(0)}(t)$ **and** $A_{(1)}(t)$ **satisfy conditions**.
  if $\deg(\Psi_1) = 0$,
    then $p := 1$
    else
      begin
        $k := 1$;
        Repeat
          $\widehat{\Psi}_{k+1}(t) := -\Psi_{k-1}(t) + H_k(t)\Psi_k(t)$
          (where $H_k(t) \in \mathbb{R}[t]$ and $\deg(\widehat{\Psi}_{k+1}) < \deg(\Psi_k)$)
          $k := k + 1$
        Until $\deg(\Psi_k) = 0$ $(k \geq 2)$
        $p := k$
      end
  $q := \begin{cases} p & \text{if } \Psi_p(t) \not\equiv 0 \\ p - 1 & \text{if } \Psi_p(t) \equiv 0 \end{cases}$
end

Fig. 1 Sturm generating algorithm (SGA)

where $\varrho(i) := \min\{k \in \mathbb{N}^* \mid \Psi_{i+k}(r) \neq 0\}$. Then, for every $t^* \in (t_a, t_b]$, we have

$$\theta_A(t^*) = \theta_A(t_a) - \arctan\{\mathcal{Q}_A(t_a)\} + \begin{cases} \arctan\{\mathcal{Q}_A(t^*)\} + [V\{\Psi(t^*)\} - V\{\Psi(t_a)\}]\pi & \text{if } A_{(0)}(t^*) \neq 0, \\ \pi/2 + [V\{\Psi(t^*)\} - V\{\Psi(t_a)\}]\pi & \text{if } A_{(0)}(t^*) = 0. \end{cases} \qquad (4)$$

2.2 Spline smoothing

Suppose that

$$\begin{aligned} &\left(\xi_k, (y_{(0)}(\xi_k), y_{(1)}(\xi_k))\right) \\ &= \left(\xi_k, (F_{(0)}(\xi_k) + \varepsilon_{(0)}(\xi_k), F_{(1)}(\xi_k) + \varepsilon_{(1)}(\xi_k))\right) \in \mathbb{R} \times \mathbb{R}^2 \end{aligned}$$

$(k = 1, 2, \ldots, n)$ are given as a sequence of 2-D noisy real vectors, where $F_{(i)} : [t_0, t_1] \to \mathbb{R}$ $(i = 0, 1)$ are unknown twice continuously differentiable functions and $\varepsilon_{(i)} : [t_0, t_1] \to \mathbb{R}$ $(i = 0, 1)$ are additive noise at $(t_0 =)\xi_1 < \xi_2 < \cdots < \xi_n (= t_1)$. If we hope to estimate $\theta_F(t^*)$ in (2), it is natural to estimate $(F_{(0)}, F_{(1)})$ first by twice continuously differentiable functions $F^*_{(i)} \in C^2[t_0, t_1]$ $(i = 0, 1)$ which are respectively the unique optimal solutions of

$$\begin{aligned} &\text{Minimize } \sum_{k=1}^{n} \left\{ \widetilde{F}_{(i)}(\xi_k) - y_{(i)}(\xi_k) \right\}^2 + \lambda \int_{t_0}^{t_1} \left\{ \widetilde{F}''_{(i)}(t) \right\}^2 dt \\ &\text{s.t. } \widetilde{F}_{(i)} \in C^2[t_0, t_1], \end{aligned} \qquad (5)$$

where $\lambda > 0$ is called a smoothing parameter controlling the trade-off between the fidelity to the data and the roughness of the solution. It can be shown [18], [21] that the optimal solution $F^*_{(i)}$ has the following properties:

(i) $F^*_{(i)}$ is a cubic polynomial, i.e., polynomial of degree 3 or less, in each interval (ξ_k, ξ_{k+1}) $(k = 1, 2 \ldots, n-1)$.

(ii) at the design point ξ_k $(k = 1, 2, \ldots, n)$, $F^*_{(i)}$ and its first two derivatives are continuous, but there may be discontinuity in the third derivative.

(iii) in the range $(-\infty, \xi_1) \cup (\xi_n, \infty)$ the second derivative is zero, so that $F^*_{(i)}$ is linear outside the range of the data.

Any function which satisfies (i) and (ii) is called a *cubic spline* with knots ξ_k $(k = 1, 2, \ldots, n)$. Any cubic spline function which satisfies (iii) is called a *natural cubic spline*. These properties are not imposed on the estimate, but arise automatically from the choice of the roughness penalty $\int_{t_0}^{t_1}\{\widetilde{F}''_{(i)}(t)\}^2 dt$. The optimal solution $F^*_{(i)}$ is called a *smoothing spline*. In particular, as $\lambda \to 0$ (no smoothing), the smoothing spline converges to the so-called interpolating spline. Moreover, as $\lambda \to \infty$ (infinite smoothing), the smoothing spline converges to a linear least squares estimate.

If a finite smoothing parameter $\lambda > 0$ is given, we can obtain the coefficients of smoothing spline $F^*_{(i)}$ by solving a band-limited linear system of size $(n-2)$ [18]. A stable and fast numerical algorithm for solving this system is available [25].

Furthermore, for robustness against impulsive noise, we can extend (4) to

$$\text{Minimize } \sum_{k=1}^{n} \rho\left(\widetilde{F}_{(i)}(\xi_k) - y_{(i)}(\xi_k)\right) + \lambda \int_{t_0}^{t_1} \left\{\widetilde{F}''_{(i)}(t)\right\}^2 dt$$
$$\text{s.t. } \widetilde{F}_{(i)} \in C^2[t_0, t_1], \tag{6}$$

by introducing a certain convex function $\rho : \mathbb{R} \to \mathbb{R}$ which is less rapidly increasing than t^2 [26], [27]. It is known that the unique optimal solution of (5) as well satisfies the above properties (i), (ii) and (iii).

Several methods for choosing the smoothing parameter λ in (4) have also been proposed, e.g., cross-validation method [20], [28].

If we use $\lambda = 0$, all cubic splines interpolating $\left(\xi_k, y_{(i)}(\xi_k)\right)$ $(k = 1, 2, \ldots, n)$ are the solution of (4) (or (5)) and not determined uniquely. A cubic interpolating spline $S(t)$ is uniquely determined if we impose additionally one of the boundary conditions below.

$$\text{(a)} \quad \begin{cases} S'(\xi_1) = F'_{(i)}(\xi_1) \\ S'(\xi_n) = F'_{(i)}(\xi_n) \end{cases} \quad \text{Hermite,}$$

$$\text{(b)} \quad \begin{cases} S''(\xi_1) = 0 \\ S''(\xi_n) = 0 \end{cases} \quad \text{Natural,}$$

$$\text{(c)} \quad \begin{cases} S'''(\xi_2-) = S'''(\xi_2+) \\ S'''(\xi_{n-1}-) = S'''(\xi_{n-1}+) \end{cases} \quad \text{Not-a-knot,}$$

$$\text{(d)} \quad \begin{cases} S'(\xi_1) = S'(\xi_n) \\ S''(\xi_1) = S''(\xi_n) \end{cases} \quad \text{Periodic.}$$

[Example 1] (Interpolating cubic spline under certain boundary conditions [22]) *A cubic spline $S(t)$, which satisfies $S(\xi_k) = y_k$ $(k = 1, 2, \ldots, n)$ and one of the above boundary conditions (a),(b),(c) and (d), is determined in each interval $[\xi_k, \xi_{k+1}]$ $(k = 1, 2, \ldots, n-1)$ as a polynomial*

$$S(t) = M_k \left\{ \frac{(\xi_{k+1} - t)^3}{6h_k} - \frac{\xi_{k+1} - t}{6} h_k \right\}$$
$$+ M_{k+1} \left\{ \frac{(t - \xi_k)^3}{6h_k} - \frac{t - \xi_k}{6} h_k \right\} + \frac{\xi_{k+1} - k}{h_k} y_k + \frac{t - \xi_k}{h_k} y_{k+1}, \tag{7}$$

with $h_k := \xi_{k+1} - \xi_k$ $(k = 1, 2, \ldots, n-1)$ and the unique solution $(M_k)_{k=1}^n$ of a system of linear equations defined by the boundary condition and

$$\mu_k M_{k-1} + 2M_k + \lambda_k M_{k+1} = d_k \quad (k = 2, 3, \ldots, n-1), \tag{8}$$

where $\mu_k := \frac{h_{k-1}}{h_{k-1}+h_k}$, $\lambda_k := \frac{h_k}{h_{k-1}+h_k}$ *and* $d_k := 6\frac{\{(y_{k+1}-y_k)/h_k\} - \{(y_k - y_{k-1})/h_{k-1}\}}{h_{k-1}+h_k}$.

In the next section, we present an extension of the algebraic phase unwrapping and propose to use spline smoothing as its preprocessing.

3. Algebraic phase unwrapping for a pair of smoothing splines

3.1 Extension of the algebraic phase unwrapping to a pair of piecewise polynomials

Suppose that $(S_{(0)}, S_{(1)})$ is a pair of piecewise real polynomials with knots ξ_k $(k = 1, 2, \ldots, n)$ s.t. $S_{(i)}(t) = A^{\langle l \rangle}_{(i)}(t) \in \mathbb{R}[t]$ $(i = 0, 1)$ in each interval $[\xi_l, \xi_{l+1}]$ $(l = 1, 2, \ldots, n-1)$. Define $\{\Psi^{\langle l \rangle}_k\}_{k=0}^{q_{\langle l \rangle}}$ be a sequence of functions obtained by applying the algorithm (SGA) in Fig. 1 to $A^{\langle l \rangle}_{(0)}(t)$ and $A^{\langle l \rangle}_{(1)}(t)$ with

$$\Psi^{\langle l \rangle}_k(t) := \frac{\widehat{\Psi}^{\langle l \rangle}_k(t)}{(t - \xi_l)^{e_{\langle l \rangle, k}}},$$

where $e_{\langle l \rangle, k}$ denotes the order of $t = \xi_l$ as a zero of polynomial $\widehat{\Psi}^{\langle l \rangle}_k(t)$. Then, if $A^{\langle l \rangle}(t) := A^{\langle l \rangle}_{(0)}(t) + jA^{\langle l \rangle}_{(1)}(t)$ satisfies $A^{\langle l \rangle}(t) \neq 0$ $(t \in [\xi_l, \xi_{l+1}])$ and $A^{\langle l \rangle}_{(0)}(\xi_l), A^{\langle l \rangle}_{(0)}(\xi_{l+1}) \neq 0$, by replacing t_a, t_b in Proposition 1 with ξ_l, ξ_{l+1} we have

$$\int_{\xi_l}^{\xi_{l+1}} \left(\arctan\left\{\frac{A^{\langle l \rangle}_{(1)}(t)}{A^{\langle l \rangle}_{(0)}(t)}\right\}\right)' dt$$
$$= \arctan\{\mathcal{Q}^{\langle l \rangle}_A(\xi_{l+1})\} - \arctan\{\mathcal{Q}^{\langle l \rangle}_A(\xi_l)\}$$
$$+ [V\{\Psi^{\langle l \rangle}(\xi_{l+1})\} - V\{\Psi^{\langle l \rangle}(\xi_l)\}]\pi, \tag{9}$$

where $\mathcal{Q}^{\langle l \rangle}_A(t) := \frac{A^{\langle l \rangle}_{(1)}(t)}{A^{\langle l \rangle}_{(0)}(t)}$. From (8) and $\arctan\{\mathcal{Q}^{\langle l \rangle}_A(\xi_{l+1})\} = \arctan\{\mathcal{Q}^{\langle l+1 \rangle}_A(\xi_{l+1})\}$, the next proposition is derived.

[Proposition 2] (Algebraic phase unwrapping for a pair of piecewise real polynomials) *Let piecewise polynomials $S_{(i)} : [\xi_1, \xi_n] \to \mathbb{R}$ $(i = 0, 1)$ satisfy $S_{(i)}(t) = A^{\langle l \rangle}_{(i)}(t) \in \mathbb{R}[t]$ in each interval $[\xi_l, \xi_{l+1}]$ $(l = 1, 2, \ldots, n-1)$, $S_{(0)}(t) + jS_{(1)}(t) \neq 0$ $(t \in [\xi_1, \xi_n])$ and $S_{(0)}(\xi_l) \neq 0$ $(l = 1, 2, \ldots, n)$ s.t. $\xi_1 < \xi_2 < \cdots < \xi_n$. For any $t^* \in (\xi_k, \xi_{k+1}]$ $(k = 1, 2, \ldots, n-1)$ we have*

$$\theta_S(t^*) = \theta_S(\xi_1) + \int_{\xi_1}^{t^*} \left(\arctan\left\{\frac{S_{(1)}(t)}{S_{(0)}(t)}\right\}\right)' dt \tag{10}$$

$$= \theta_S(\xi_1) + \sum_{l=1}^{k-1} \int_{\xi_l}^{\xi_{l+1}} \left(\arctan\left\{\frac{A^{\langle l \rangle}_{(1)}(t)}{A^{\langle l \rangle}_{(0)}(t)}\right\}\right)' dt$$
$$+ \int_{\xi_k}^{t^*} \left(\arctan\left\{\frac{A^{\langle k \rangle}_{(1)}(t)}{A^{\langle k \rangle}_{(0)}(t)}\right\}\right)' dt \tag{11}$$

$$= \theta_S(\xi_1) - \arctan\{\mathcal{Q}^{\langle 1 \rangle}_A(\xi_1)\}$$
$$+ \sum_{l=1}^{k-1} [V\{\Psi^{\langle l \rangle}(\xi_{l+1})\} - V\{\Psi^{\langle l \rangle}(\xi_l)\}]\pi$$
$$+ \begin{cases} \arctan\{\mathcal{Q}^{\langle k \rangle}_A(t^*)\} + [V\{\Psi^{\langle k \rangle}(t^*)\} - V\{\Psi^{\langle k \rangle}(\xi_k)\}]\pi \\ \qquad\qquad \text{if } A^{\langle k \rangle}_{(0)}(t^*) \neq 0, \\ \pi/2 + [V\{\Psi^{\langle k \rangle}(t^*)\} - V\{\Psi^{\langle k \rangle}(\xi_k)\}]\pi \\ \qquad\qquad \text{if } A^{\langle k \rangle}_{(0)}(t^*) = 0, \end{cases} \tag{12}$$

$$= \theta_S(\xi_k) - \arctan\{\mathcal{Q}_A^{\langle k\rangle}(\xi_k)\} + \begin{cases} \arctan\{\mathcal{Q}_A^{\langle k\rangle}(t^*)\} + [V\{\Psi^{\langle k\rangle}(t^*)\} - V\{\Psi^{\langle k\rangle}(\xi_k)\}]\pi & \text{if } A_{(0)}^{\langle k\rangle}(t^*) \neq 0, \\ \pi/2 + [V\{\Psi^{\langle k\rangle}(t^*)\} - V\{\Psi^{\langle k\rangle}(\xi_k)\}]\pi & \text{if } A_{(0)}^{\langle k\rangle}(t^*) = 0. \end{cases} \quad (13)$$

3.2 Spline smoothing as preprocessing of algebraic phase unwrapping

Let $S_{(i)}$ $(i = 0, 1)$ be the optimal solutions of problem (4) or (5). Although the unwrapped phase is a highly non-linear functional of $(F_{(0)}, F_{(1)})$, we propose to use $\theta_S(t^*)$ as an estimate of $\theta_F(t^*)$ in a way similar to an estimation technique [25] where $S'_{(i)}$ is used as the estimate of $F'_{(i)}$. To summarize the proposed phase unwrapping technique, first we compute two natural cubic splines $S_{(0)}$ and $S_{(1)}$ as the optimal solutions to the problem (4) or (5). $S_{(0)}$ and $S_{(1)}$ approximate respectively $F_{(0)}$ and $F_{(1)}$. Second, by applying the extended algebraic phase unwrapping in Proposition 2 to the pair of smoothing splines, we obtain $\theta_S(t^*)$ which would be a good estimate of $\theta_F(t^*)$.

4. Numerical Examples

4.1 Phase unwrapping for noise-free data

In this section, we examine the performance of the proposed phase unwrapping technique for a pair of two univariate real functions $f_{(0)}, f_{(1)}$ and $\Upsilon(t) = t$ which satisfy the condition (1). In this case, we have $F_{(0)}(t) := f_{(0)}(\Upsilon(t)) = f_{(0)}(t)$ and $F_{(1)}(t) := f_{(1)}(\Upsilon(t)) = f_{(1)}(t)$. We compare the proposed technique and a standard phase unwrapping algorithm [29] which computes an estimate of $\theta_F(\xi_k)$ as

$$\widetilde{\theta}_F(\xi_k) := \theta_F(\xi_1) + \sum_{l=0}^{k-1} \mathcal{W}(\mathcal{W}(\theta_F(\xi_{l+1})) - \mathcal{W}(\theta_F(\xi_l))), \quad (14)$$

where $\mathcal{W}(x) := x + 2\pi k(x)$ s.t. $k(x)$ is a integer chosen to satisfy $-\pi < \mathcal{W}(x) \leq \pi$ (i.e., $\mathcal{W}(\xi_l)$ satisfies $\mathcal{W}(\xi_l) \in (-\pi, \pi]$ and $F_{(0)}(\xi_l) + jF_{(1)}(\xi_l) = |F_{(0)}(\xi_l) + jF_{(1)}(\xi_l)|e^{j\mathcal{W}(\theta_F(\xi_l))}$). $\mathcal{W}(\theta_F(\xi_l))$ is called the wrapped phase of $(f_{(0)}, f_{(1)})$ at $t = \xi_l$.

The standard phase unwrapping is designed based on a strong assumption:

$$-\pi < \theta_F(\xi_{l+1}) - \theta_F(\xi_l) \leq \pi \quad (l = 1, 2, \ldots, n-1).$$

[Example 2] *Suppose that $\theta_F(t)$, $f_{(0)}$ and $f_{(1)}$ are defined respectively as*

$$\theta_F(t) := -\frac{5}{5 + (t-5)^2} + \frac{1}{200}t^3 - \frac{1}{20}t^2 + \frac{1}{10}t + \frac{1}{6} + \sin\left(\frac{\pi}{2}t\right),$$

$F_{(0)}(t) = f_{(0)}(t) := \cos(\theta_F(t))$ *and* $F_{(1)}(t) = f_{(1)}(t) := \sin(\theta_F(t))$. *A data sequence of* 2*-D real vectors are generated at* $\xi_k = (k-1)h$ *as* $\left(\xi_k, (F_{(0)}(\xi_k), F_{(1)}(\xi_k))\right)$ $(k = 1, 2, \ldots, n)$. *The spline functions $S_{(0)}$ and $S_{(1)}$ are defined as the optimal solutions of problem* (4) *for $\lambda \to 0$, i.e., $S_{(0)}$ and $S_{(1)}$ are interpolating natural cubic splines. Fig. 2 and Fig. 3 demonstrate the performance of the phase unwrapping algorithms where the small circles indicate points sampled at intervals of $h = 0.4$ in Fig. 2 and $h = 0.55$ in Fig. 3.*

In Fig. 2(a) and Fig. 3(a), the dashed line depicts $F_{(0)}(t) = \cos(\theta_F(t))$ and the solid line depicts the spline function $S_{(0)}(t)$. In Fig. 2(b) and Fig. 3(b), the dashed line depicts $F_{(1)}(t) = \sin(\theta_F(t))$ and the solid line depicts the spline function $S_{(1)}(t)$. In Fig. 2(c) and Fig. 3(c), the dashed line depicts $\theta_F(t)$, the short dashed line depicts the estimate $\widetilde{\theta}_F(t)$ obtained by the standard phase unwrapping (13) *and the solid line depicts the estimate $\theta_S(t)$ achieved by the proposed phase unwrapping.*

From Fig. 2, we observe that the standard algorithm and the proposed phase unwrapping technique estimate exactly $\theta_F(t)$ at all sampling points if sampling interval h is small enough.

From Fig. 3, we observe that the proposed phase unwrapping technique estimates $\theta_F(t)$ exactly at all sampling points even if the sampling interval h is not small enough for the standard algorithm.

4.2 Phase unwrapping for noisy data

[Example 3] *Suppose that $\theta_F(t)$, $f_{(0)}$ and $f_{(1)}$ are defined as Example 2. A data sequence of* 2*-D noisy real vectors are generated at $\xi_k = (k-1)h$ $(h = 0.5)$ as*

$$\left(\xi_k, (y_{(0)}(\xi_k), y_{(1)}(\xi_k))\right) = \left(\xi_k, (F_{(0)}(\xi_k) + \varepsilon_{(0)}(\xi_k), F_{(1)}(\xi_k) + \varepsilon_{(1)}(\xi_k))\right),$$

where $\varepsilon_{(i)}$ $(i = 0, 1)$ is gaussian noise $\mathcal{N}(0, 1/25)$.

From Fig. 4, we observe that the proposed phase unwrapping technique estimates $\theta_F(t)$ exactly at all sampling points even if the noise is not small enough for the standard algorithm.

5. Conclusion

The proposed phase unwrapping technique is designed as a natural combination of the extended algebraic phase unwrapping and the spline smoothing for a given sequence of 2-D real vectors. Numerical examples demonstrate the effectiveness of the proposed technique.

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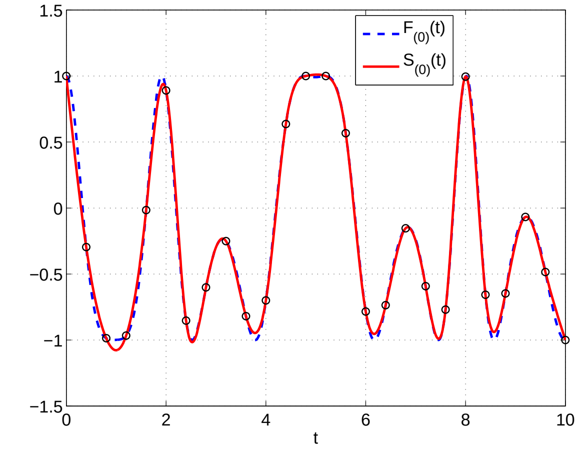
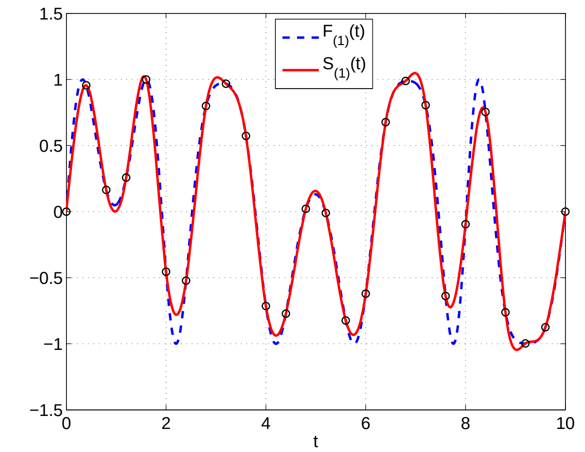
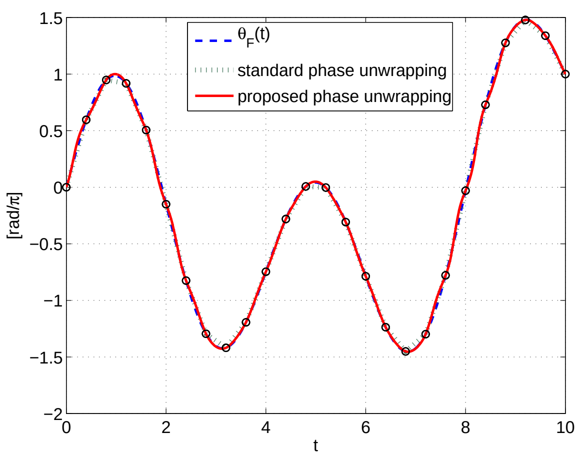


(a) $F_{(0)}(t) = f_{(0)}(t) = \cos(\theta_F(t))$ and smoothing spline $S_{(0)}(t)$

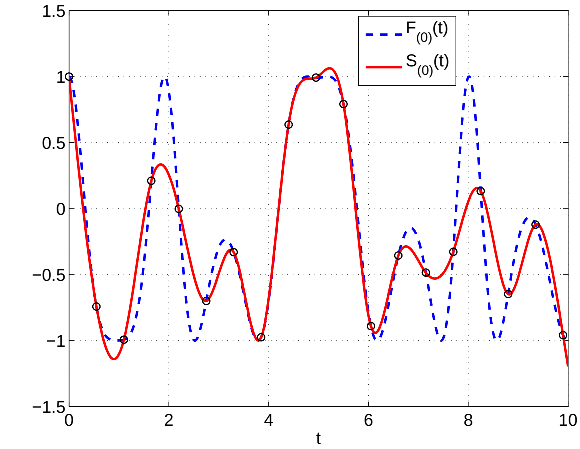


(b) $F_{(1)}(t) = f_{(1)}(t) = \sin(\theta_F(t))$ and smoothing spline $S_{(1)}(t)$

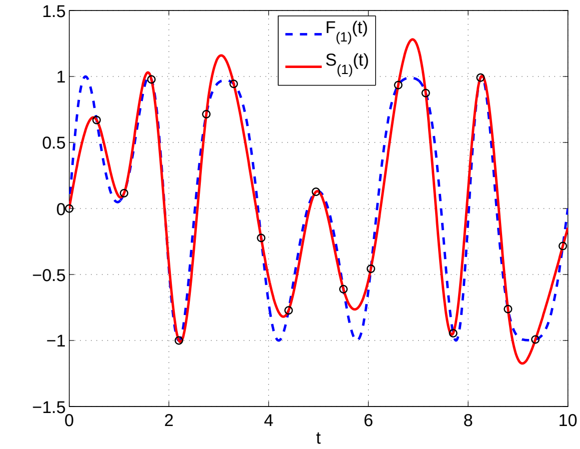


(c) $\theta_F(t)$ and estimates of $\theta_F(t)$

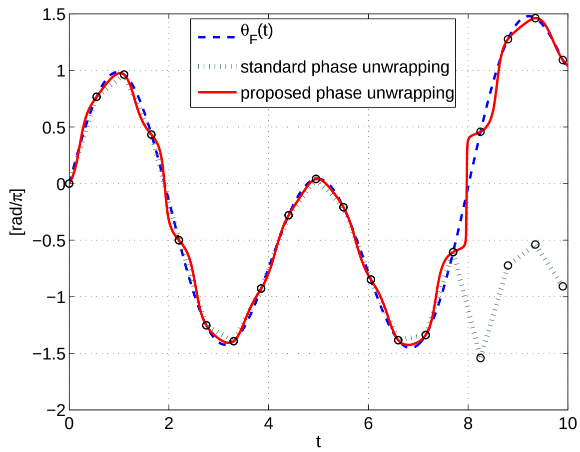
Fig. 2 Phase unwrapping for $(f_{(0)}, f_{(1)})$ using noise-free data ($h = 0.4$)



(a) $F_{(0)}(t) = f_{(0)}(t) = \cos(\theta_F(t))$ and smoothing spline $S_{(0)}(t)$

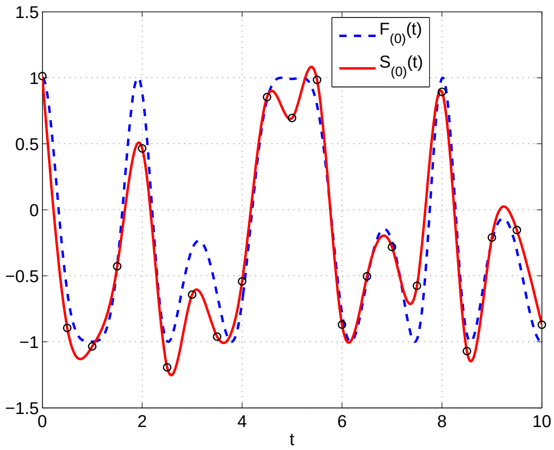


(b) $F_{(1)}(t) = f_{(1)}(t) = \sin(\theta_F(t))$ and smoothing spline $S_{(1)}(t)$

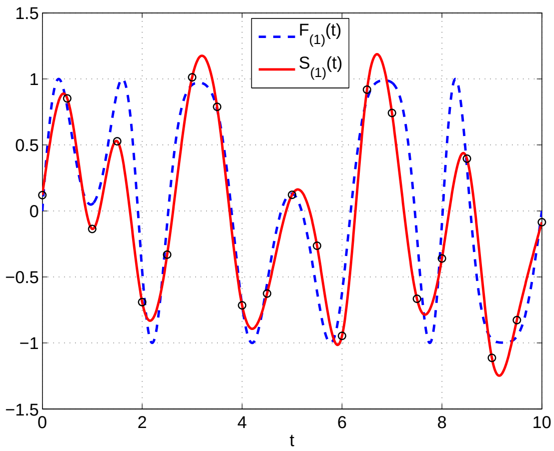


(c) $\theta_F(t)$ and estimates of $\theta_F(t)$

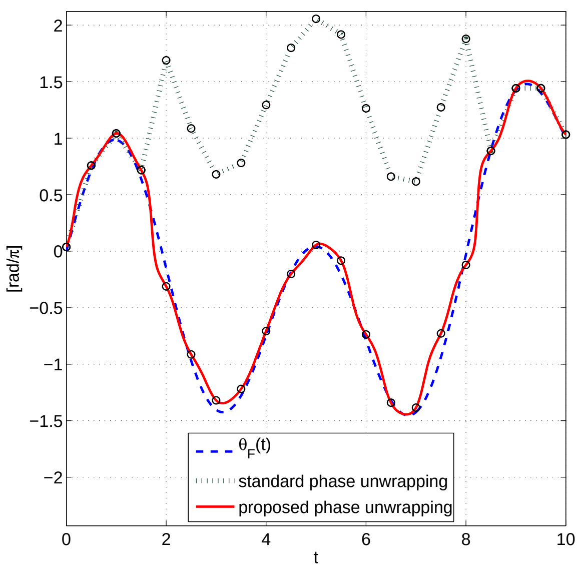
Fig. 3 Phase unwrapping for $(f_{(0)}, f_{(1)})$ using noise-free data ($h = 0.55$)



(a) $F_{(0)}(t) = f_{(0)}(t) = \cos(\theta_F(t))$ and smoothing spline $S_{(0)}(t)$



(b) $F_{(1)}(t) = f_{(1)}(t) = \sin(\theta_F(t))$ and smoothing spline $S_{(1)}(t)$



(c) $\theta_F(t)$ and estimates of $\theta_F(t)$

Fig. 4 Phase unwrapping for $(f_{(0)}, f_{(1)})$ using noisy data ($h = 0.5$)

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