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Mixed Trigonometric Interpolation Techniques for Fast and Stable Algebraic Phase Unwrapping

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Abstract The algebraic phase unwrapping was established in 1998 as a rigorous symbolic algebraic solution to the phase unwrapping problem, i.e., the problem of computing the continuous phase function of a given complex polynomial. In this paper, we propose a simple but a powerful numerical stabilization technique named *the mixed trigonometric interpolation* for the algebraic phase unwrapping. This technique is based on replacing a certain set of arithmetic operations in polynomial ring by an interpolation of a certain mixed trigonometric function. By this technique we can obtain numerical stable approximation of *general Sturm sequence* without suffering the coefficient growth. Moreover, by combining *the mixed trigonometric interpolation* with FFT, we succeeded in making *the mixed trigonometric interpolation* faster and more stable. The proposed techniques allow us to solve phase unwrapping problem along the unit circle stably even if the degree of a given polynomial is very large.

Keywords Algebraic phase unwrapping, Numerical stabilization, Coefficient growth, General Sturm sequence, Mixed trigonometric interpolation, FFT.

1. Introduction

For a given complex polynomial $A(z) \in \mathbb{C}[z]$ such that

$$\left.\begin{array}{ll} A(z) \neq 0 & \text{for all } |z| = 1 \\ \Re\{A(1)\} \neq 0 & \end{array}\right\}, \qquad (1)$$

the phase unwrapping along the unit circle is a problem to compute for $\omega^* \in (0, 2\pi]$

$$\begin{aligned} \theta_A(\omega^*) &= \theta_A(0) + \int_0^{\omega^*} \Im\left\{\frac{A'(e^{j\omega})}{A(e^{j\omega})}\right\} d\omega \\ &= \theta_A(0) + \int_0^{\omega^*} \left(\arctan\left\{\frac{\Im\{A(e^{j\omega})\}}{\Re\{A(e^{j\omega})\}}\right\}\right)' d\omega, \quad (2) \end{aligned}$$

where $-\pi < \theta_A(0) \leq \pi$ and $A(1) = |A(1)|e^{j\theta_A(0)}$. A rigorous symbolic algebraic solution to the phase unwrapping problem was established in 1998 [1] (see Proposition 1 in Sect. 2. 2). This method does not require any numerical root finding or numerical integration technique.

Recently, it was shown in [2] that the algebraic phase unwrapping can be applied as a powerful mathematical tool to compute the *Minimum-Maximum distributions of self-reciprocal Laurent polynomial along the unit circle*, which implies that the algorithm can be applied to the estimation of *the Directions-of-Arrival distribution* (DOA distribution): the number of directions of signals in an arbitrarily specified range, which is a valuable information in many array signal processing applications [3] in analogy with the idea of the MUSIC algorithm [4].

However, in a direct computer implementation of the algebraic phase unwrapping algorithm (SGA) in Fig. 1 for polynomials of large degree, we encounter certain serious instabilities due to the unavoidable gap between numerical value computed by digital computer and theoretical value. Therefore, thoughtless direct computation of SGA for polynomials of large degree, often results in the failure of a key property of the desired *general Sturm sequence*, which is generated by applying SGA, leading thus the failure of the exact phase unwrapping in the end.

In this paper, we propose a simple but a powerful numerical stabilization technique named *the mixed trigonometric interpolation* for the algebraic phase unwrapping. The proposed stabilization technique produces a good approximation of the ideal *general Sturm sequence* by reducing the inductive step in SGA to a certain system of linear equations. Moreover, by combining *the mixed trigonometric interpolation* with FFT, we make *the mixed trigonometric interpolation* faster and more stable. Thanks to the key property of the approximation, of *general Sturm sequence*, guaranteed by the proposed stabilization techniques, the algebraic phase unwrapping is stabilized greatly even for polynomials of large degree, which implies that the algebraic phase unwrapping is applicable to practical array signal processing problems formulated with polynomials of large degree.

2. Preliminaries

2.1 Notation

Let $\mathbb{C}$ denote the set of all complex numbers. We use $j \in \mathbb{C}$ to denote the imaginary unit satisfying $j^2 = -1$. For any $c \in \mathbb{C}$, $\Re(c)$, $\Im(c)$ and $\bar{c}$ stand respectively for the real part, the imaginary part, and complex conjugate of c. For any $C(z) = \sum_{k=l}^{m} c_k z^k \in \mathbb{C}[z, z^{-1}]$ (s.t. $c_l c_m \neq 0$ and $l \leq m$), we define $\deg(C) := m$, $\mathrm{l}\deg(C) := l$, $\mathrm{c}\deg(C) := \frac{l+m}{2}$, $C^*(z) := \sum_{k=l}^{m} \bar{c}_{m-k+l} z^k$, $C_F(\omega) := C(e^{j\omega})$, $C^\dagger(z) := z^{-\mathrm{c}\deg(C)} C(z) \in \mathbb{C}[z^{1/2}, z^{-1/2}]$, $C_F^\dagger(\omega) := C^\dagger(e^{j\omega})$, $C_{(0)}(z) := \frac{C(z)+C^*(z)}{2}$ and $C_{(1)}(z) := \frac{C(z)-C^*(z)}{2j}$. The degrees of the constant 0 are defined as $\deg(0) = \mathrm{l}\deg(0) = \mathrm{c}\deg(0) = 0$. In particular, $C(z) \in \mathbb{C}[z, z^{-1}]$ satisfying $C(z) = C^*(z)$ is called a self-reciprocal Laurent polynomial. If $C(z)$ is self-reciprocal, $C^\dagger(z)$ and $C_F^\dagger(\omega)$ are expressed as follows,

$$C^\dagger(z) = \begin{cases} c_{\frac{l+m}{2}} + \displaystyle\sum_{k=1}^{\frac{m-l}{2}} \left(c_{\frac{l+m+2k}{2}} z^k + \bar{c}_{\frac{l+m+2k}{2}} z^{-k} \right) \\ \qquad \left(\text{s.t. } c_{\frac{l+m}{2}} \in \mathbb{R}\right) \qquad \text{if } (l+m) \text{ is even,} \\ \displaystyle\sum_{k=1}^{\frac{m-l+1}{2}} \left(c_{\frac{l+m+2k-1}{2}} z^{\frac{2k-1}{2}} + \bar{c}_{\frac{l+m+2k-1}{2}} z^{-\frac{2k-1}{2}} \right) \\ \qquad \text{if } (l+m) \text{ is odd,} \end{cases} \tag{3}$$

$$C_F^\dagger(\omega) = \begin{cases} c_{\frac{l+m}{2}} + 2\displaystyle\sum_{k=1}^{\frac{m-l}{2}} \left\{ \Re\left(c_{\frac{l+m+2k}{2}}\right) \cos k\omega \right. \\ \qquad \left. - \Im\left(c_{\frac{l+m+2k}{2}}\right) \sin k\omega \right\} \qquad \text{if } (l+m) \text{ is even,} \\ 2\displaystyle\sum_{k=1}^{\frac{m-l+1}{2}} \left\{ \Re\left(c_{\frac{l+m+2k-1}{2}}\right) \cos \tfrac{2k-1}{2}\omega \right. \\ \qquad \left. - \Im\left(c_{\frac{l+m+2k-1}{2}}\right) \sin \tfrac{2k-1}{2}\omega \right\} \quad \text{if } (l+m) \text{ is odd.} \end{cases} \tag{4}$$

Obviously, $C_F^\dagger(\omega)$ is a real-valued differentiable function over $\mathbb{R}$. For any $C(z) \in \mathbb{C}[z, z^{-1}]$, we have $C(z) = C_{(0)}(z) + jC_{(1)}(z)$, where $C_{(0)}(z)$ and $C_{(1)}(z)$ are self-reciprocal, and $\mathrm{c}\deg(C) = \mathrm{c}\deg(C_{(0)}) = \mathrm{c}\deg(C_{(1)})$. Moreover, we have

$$\left.\begin{aligned} \Re\{C_F^\dagger(\omega)\} &= \Re\{C_{(0)F}^\dagger(\omega) + jC_{(1)F}^\dagger(\omega)\} = C_{(0)F}^\dagger(\omega) \\ \Im\{C_F^\dagger(\omega)\} &= \Im\{C_{(0)F}^\dagger(\omega) + jC_{(1)F}^\dagger(\omega)\} = C_{(1)F}^\dagger(\omega) \end{aligned}\right\}. \tag{5}$$

2.2 Algebraic phase unwrapping

The next proposition is a slight generalization of the main theorem in [1]. This proposition enables us to solve the phase unwrapping problem in symbolic algebraic ways.

[Proposition 1] (Algebraic phase unwrapping along the unit circle [2]) *Let* $A(z) := A_{(0)}(z) + jA_{(1)}(z) \in \mathbb{C}[z]$, *where* $A_{(0)}(z), A_{(1)}(z) \in \mathbb{C}[z]$ *are self-reciprocal polynomials satisfying* $\mathrm{c}\deg(A) = \mathrm{c}\deg(A_{(0)}) = \mathrm{c}\deg(A_{(1)})$, $A_{(0)}(1) \neq 0$ *and* $A(z) \neq 0$ *for* $|z| = 1$. *Define*

$$\begin{aligned} \mathcal{Z}_{A_{(0)}}^\dagger &:= \{\omega \in [0, 2\pi] \mid A_{(0)F}^\dagger(\omega) = 0\} \\ &= \begin{cases} \emptyset & \text{if } A_{(0)F}^\dagger(\omega) \neq 0 \text{ for all } \omega \in [0, 2\pi], \\ \{\nu_1, \nu_2, \ldots, \nu_r\} & \text{otherwise,} \end{cases} \end{aligned}$$

where $\nu_0 (:=) 0 < \nu_1 < \cdots < \nu_r < 2\pi$, *and*

$$\mathcal{X}(\nu_i) := \begin{cases} +1 \\ \text{if} \begin{cases} A_{(0)F}^\dagger(\omega) A_{(1)F}^\dagger(\omega) > 0 & \text{for } \omega \in (\nu_i - \varepsilon, \nu_i) \text{ and} \\ A_{(0)F}^\dagger(\omega) A_{(1)F}^\dagger(\omega) < 0 & \text{for } \omega \in (\nu_i, \nu_i + \varepsilon), \end{cases} \\ -1 \\ \text{if} \begin{cases} A_{(0)F}^\dagger(\omega) A_{(1)F}^\dagger(\omega) < 0 & \text{for } \omega \in (\nu_i - \varepsilon, \nu_i) \text{ and} \\ A_{(0)F}^\dagger(\omega) A_{(1)F}^\dagger(\omega) > 0 & \text{for } \omega \in (\nu_i, \nu_i + \varepsilon), \end{cases} \\ 0 \ \text{otherwise,} \end{cases}$$

for ν_i $(i = 1, 2, \ldots, r)$ *and for sufficiently small* $\varepsilon > 0$. *Then we have the following relations.*

(a) *For any* $\omega^* \in (0, 2\pi]$,

$$\begin{aligned} \theta_A(\omega^*) &= \theta_A(0) + \int_0^{\omega^*} \left(\arctan\left\{ \frac{\Im\{A_F(\omega)\}}{\Re\{A_F(\omega)\}} \right\} \right)' d\omega \\ &= \theta_A(0) + \mathrm{c}\deg(A)\omega^* - \arctan\{Q_A^\dagger(0)\} \\ &\quad + \lim_{\nu \to \omega^* - 0} \arctan\{Q_A^\dagger(\nu)\} + \Lambda(\omega^*)\pi, \end{aligned} \tag{6}$$

where $Q_A^\dagger(\omega) := \frac{A_{(1)F}^\dagger(\omega)}{A_{(0)F}^\dagger(\omega)}$ *and* $\Lambda(\omega^*) := \sum_{\nu_i \in (0, \omega^*)} \mathcal{X}(\nu_i)$.

(b) *Let* $\{\Phi_k(\omega)\}_{k=0}^q$ *be a sequence of functions over* $0 \leq \omega \leq 2\pi$ *obtained by applying the algorithm (SGA) in Fig. 1 to* $A_{(0)}(z)$ *and* $A_{(1)}(z)$ *with*

$$\left.\begin{aligned} D_k(z) &:= z^{-\mathrm{l}\deg(A_{(k)})} \left(\tfrac{j}{z-1}\right)^{e_k} A_{(k)}(z) \\ \Phi_k(\omega) &:= D_{kF}^\dagger(\omega) \end{aligned}\right\}, \tag{7}$$

where e_k *denotes the order of* $z = 1$ *as a zero of polynomial* $A_{(k)}(z)$. *Define for each* $\omega \in [0, 2\pi]$ *the number of variations in the sign of* $\{\Phi_k(\omega)\}_{k=0}^q$ *by*

$$\begin{aligned} V\{\Phi(\omega)\} &:= V\{\Phi_0(\omega), \Phi_1(\omega), \ldots, \Phi_q(\omega)\} \\ &:= \left|\{i \mid 0 \leq i < q \text{ and } \Phi_i(\omega)\Phi_{i+\varrho(i)}(\omega) < 0\}\right|, \end{aligned} \tag{8}$$

where $\varrho(i) := \min\{k \in \mathbb{N}^* \mid \Phi_{i+k}(\omega) \neq 0\}$. *Then, for every* $\omega^* \in (0, 2\pi]$, *we have*

$$\begin{aligned} \theta_A(\omega^*) &= \theta_A(0) + \mathrm{c}\deg(A)\omega^* - \arctan\{Q_A^\dagger(0)\} \\ &+ \begin{cases} \arctan\{Q_A^\dagger(\omega^*)\} + [V\{\Phi(\omega^*)\} - V\{\Phi(0)\}]\pi \\ \qquad \text{if } A_{(0)F}^\dagger(\omega^*) \neq 0, \\ \pi/2 + [V\{\Phi(\omega^*)\} - V\{\Phi(0)\}]\pi \\ \qquad \text{if } A_{(0)F}^\dagger(\omega^*) = 0. \end{cases} \end{aligned} \tag{9}$$

In this paper, we call the sequence of functions $\{\Phi_k(\omega)\}_{k=0}^q$ in (7) *general Sturm sequence*.

[Example 1] *By using the SGA and Proposition 1, let us construct the unwrapped phase of the univariate complex polynomial*

$$A(z) := (6 - 4j)z^4 + (8 - 2j)z - (18 + 12j),$$

which satisfies (1). *Then,* $A_{(0)}(z)$ *and* $A_{(1)}(z)$ *are respectively*

begin
 Let $A_{(0)}(z)$ **and** $A_{(1)}(z)$ **satisfy conditions**.
 if $\deg(D_1) = 0$,
  then $p := 1$
 else
  begin
   $k := 1$;
   Repeat
    $v_k := \deg(D_{k-1}) - \deg(D_k)$
    $\beta_k := \frac{D_{k-1}(0)}{D_k(0)}$, $\gamma_k := j^{1-v_k}\beta_k$

$$A_{(k+1)}(z) := \begin{cases} (\beta_k + \bar{\beta}_k z^{v_k})D_k(z) - D_{k-1}(z) & \text{if } v_k > 0, \\ (\gamma_k + \bar{\gamma}_k z)D_k(z) - (\frac{z-1}{j})^{1-v_k} D_{k-1}(z) & \text{if } v_k \le 0, \end{cases}$$

    $k := k+1$
   Until $\deg(D_k) = 0$ $(k \ge 2)$
   $p := k$
  end

$$q := \begin{cases} p & \text{if } D_p(z) \not\equiv 0 \\ p-1 & \text{if } D_p(z) \equiv 0 \end{cases}$$

end;

Fig. 1 Sturm generating algorithm (SGA)

$$\begin{aligned} A_{(0)}(z) &= \frac{A(z) + A^*(z)}{2} \\ &= -(6-4j)z^4 + (4+j)z^3 + (4-j)z - (6+4j), \\ A_{(1)}(z) &= \frac{A(z) - A^*(z)}{2j} \\ &= -(8+12j)z^4 - (1-4j)z^3 - (1+4j)z - (8-12j). \end{aligned}$$

Applying SGA to $A_{(0)}(z)$ and $A_{(1)}(z)$, we obtain general Sturm sequence $\{\Phi_k(\omega)\}_{k=0}^5$ by

$$\begin{aligned} \Phi_0(\omega) &= -12\cos 2\omega - 8\sin 2\omega + 8\cos\omega - 2\sin\omega, \\ \Phi_1(\omega) &= -16\cos 2\omega + 24\sin 2\omega - 2\cos\omega - 8\sin\omega, \\ \Phi_2(\omega) &= 15\cos\frac{3}{2}\omega - 28\sin\frac{3}{2}\omega + 3\cos\frac{\omega}{2} + 12\sin\frac{\omega}{2}, \\ \Phi_3(\omega) &= -\frac{10878}{1009}\cos\omega + \frac{23720}{1009}\sin\omega - \frac{3792}{1009}, \\ \Phi_4(\omega) &= \frac{3031140705948}{171774501889}\cos\frac{\omega}{2} - \frac{7054071488152}{171774501889}\sin\frac{\omega}{2}, \\ \Phi_5(\omega) &= -\frac{218427060633001207927724246\,94}{371739176162917730525402451\,7}. \end{aligned}$$

The unwrapped phase $\theta_A(\omega)$ over $[0, 2\pi]$ is depicted in Fig. 2.

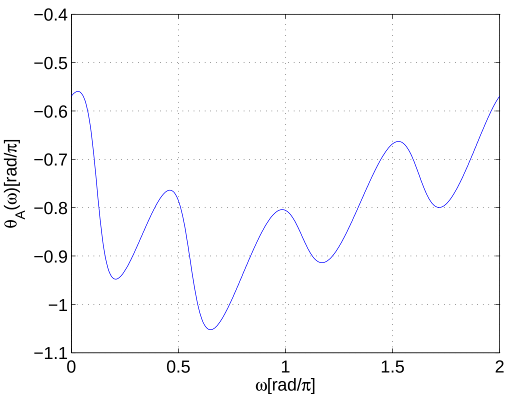

Fig. 2 Exact unwrapped phase by Proposition 1 (Example 1)

2.3 Numerical instabilities of SGA

To implement the algorithm (SGA) in Fig. 1 precisely, we need large number of digits to express the rational coefficients of the polynomials $A_{(k)}(z)$ or $D_k(z)$ and functions $\Phi_k(\omega)$ $(k = 0, 1, \ldots, q)$ mainly due to the repeated computations of β_k $(k = 1, 2, \ldots, q-1)$ (e.g., see Example 1). We call this phenomenon the *coefficient growth* in analogy with the typical cases in the computation of the *standard Sturm sequence* through the Euclid's algorithm [5]. In computer implementation of $\theta_A(\omega)$ in (9) through SGA, we encounter certain serious instabilities due to, e.g., (i) the truncation error of the trigonometric function values, and (ii) the coefficient growth which causes the truncation error in the floating-point expression of the rational coefficients (or memory shortages by increasing number of digits for exact expression of the rational coefficients). As a result, thoughtless direct computation of SGA often results in the failure of *general Sturm sequence's* key property:

$$\Phi_k(\omega_0) = 0 \text{ at } \omega_0 \in [0, 2\pi] \Rightarrow \Phi_{k-1}(\omega_0)\Phi_{k+1}(\omega_0) < 0 \quad \text{for } 0 < k < q(\ge 2), \tag{10}$$

leading thus the failure of (9). This situation restricts the practical applicability of Proposition 1 especially for polynomials $A(z) \in \mathbb{C}[z]$ of large degree.

In the next section, we propose a simple but powerful stabilization technique to maintain the key property (10) for exact phase unwrapping.

3. Mixed Trigonometric Interpolation for Stabilization of Algebraic Phase Unwrapping

For numerical implementation of the algebraic phase unwrapping through SGA, we need a certain stable approximation of the value $V\{\Phi(\omega)\}$. This goal is achieved by a careful numerical approximation $\{\widetilde{\Phi}_k(\omega)\}_{k=0}^q (\approx \{\Phi_k(\omega)\}_{k=0}^q)$ which is guaranteed to satisfy

$$\begin{cases} \widetilde{\Phi}_0 = \Phi_0, \widetilde{\Phi}_1 = \Phi_1, \\ \widetilde{\Phi}_k(\omega_0) = 0 \text{ at } \omega_0 \in [0, 2\pi] \Rightarrow \widetilde{\Phi}_{k-1}(\omega_0)\widetilde{\Phi}_{k+1}(\omega_0) < 0 \\ \qquad \text{for } 0 < k < q(\ge 2). \end{cases} \tag{11}$$

Indeed by (11), we can obtain $V\{\widetilde{\Phi}(\omega)\} - V\{\widetilde{\Phi}(0)\} (= V\{\Phi(\omega)\} - V\{\Phi(0)\})$ unless ω is in the vicinity of zeros of Φ_0. (Note: The unavoidable gap between numerical and theoretical zeros of Φ_0 does not guarantee $V\{\widetilde{\Phi}(\omega)\} - V\{\widetilde{\Phi}(0)\} = V\{\Phi(\omega)\} - V\{\Phi(0)\}$ for ω in the vicinity of such zeros). The proposed construction of $\{\widetilde{\Phi}_k(\omega)\}_{k=0}^q (\approx \{\Phi_k(\omega)\}_{k=0}^q)$ is presented inductively as follows.

Suppose that we have $m \ge 1, n \ge 1$ and

$$\left.\begin{aligned} \widetilde{\Phi}_{k-1}(\omega) &= a_m \cos\tfrac{m}{2}\omega + a_{m-1}\sin\tfrac{m}{2}\omega + a_{m-2}\cos\tfrac{m-2}{2}\omega + \cdots \\ \widetilde{\Phi}_k(\omega) &= b_n \cos\tfrac{n}{2}\omega + b_{n-1}\sin\tfrac{n}{2}\omega + b_{n-2}\cos\tfrac{n-2}{2}\omega + \cdots \end{aligned}\right\}.$$

Under the standard assumption,(1) we can express $\widetilde{\Phi}_{k+1}(\omega)$ for

(1): Almost always, we can assume (i) $A_{(k)}(1) \ne 0$ $(k = 2, 3, \ldots, q)$ and (ii) $\left.\begin{aligned}\deg(D_{k-1}) &= m \\ \deg(D_k) &= n\end{aligned}\right\} \Rightarrow \deg(D_{k+1}) = \begin{cases} m-2 & \text{if } m > n, \\ n-1 & \text{if } m \le n. \end{cases}$

$m > n$ by

$\widetilde{\Phi}_{k+1}(\omega) = c_{m-2}\cos\frac{m-2}{2}\omega + c_{m-3}\sin\frac{m-2}{2}\omega + c_{m-4}\cos\frac{m-4}{2}\omega + \cdots$,

for $m \leq n$ by

$\widetilde{\Phi}_{k+1}(\omega) = c_{n-1}\cos\frac{n-1}{2}\omega + c_{n-2}\sin\frac{n-1}{2}\omega + c_{n-3}\cos\frac{n-3}{2}\omega + \cdots$.

To determine $\{c_i\}_{i=0}^{m-2}$ (or $\{c_i\}_{i=0}^{n-1}$) without suffering the coefficient growth in the direct computation of SGA, we reduce this problem to an interpolation problem of the mixed trigonometric function $\widetilde{\Phi}_{k+1}(\omega)$. By choosing carefully sample points ω_i ($i = 0, 1 \ldots, m-2$ (or $n-1$)) for this interpolation problem, we can determine $\{c_i\}_{i=0}^{m-2}$ (or $\{c_i\}_{i=0}^{n-1}$) uniquely by solving a system of linear equations.

Applying the idea in SGA to $\widetilde{\Phi}_{k-1}(\omega)$ and $\widetilde{\Phi}_k(\omega)$, we found that, in order to ensure (11), $\widetilde{\Phi}_{k+1}(\omega)$ is desired to approximate $\widetilde{\Psi}_{k+1}(\omega)$ for all $\omega \in [0, 2\pi]$, where for $m > n$

$$\widetilde{\Psi}_{k+1}(\omega) := \frac{2\widetilde{\Phi}_k(\omega)}{b_n^2 + b_{n-1}^2}\left\{(a_m b_n + a_{m-1}b_{n-1})\cos\frac{m-n}{2}\omega - (a_m b_{n-1} - a_{m-1}b_n)\sin\frac{m-n}{2}\omega\right\} - \widetilde{\Phi}_{k-1}(\omega), \quad (12)$$

for $m \leq n$

$$\widetilde{\Psi}_{k+1}(\omega) := \frac{2\widetilde{\Phi}_k(\omega)}{b_n^2 + b_{n-1}^2}\left\{(a_m b_n + a_{m-1}b_{n-1})\cos\frac{\omega-(1+n-m)\pi}{2} - (a_m b_{n-1} - a_{m-1}b_n)\sin\frac{\omega-(1+n-m)\pi}{2}\right\} - \widetilde{\Phi}_{k-1}(\omega)\sum_{l=0}^{1+n-m}\binom{1+n-m}{l}\cos\frac{(1+n-m-2l)(\pi-\omega)}{2}. \quad (13)$$

In the following construction of $\widetilde{\Phi}_{k+1}(\omega)$, we assume for simplicity the case $m > n$. (Note: The discussion for the other case is almost same). Unfortunately, thoughtless direct computation of $\{c_i\}_{i=0}^{m-2}$ through the elementary trigonometric expansion of equation (12) causes the coefficient growth as remarked in Sect. 2.3.

We take a different path to find a stable numerical approximation of $\{c_i\}_{i=0}^{m-2}$ by using the relation (12) for numerical evaluation of $\widetilde{\Psi}_{k+1}(\omega_i)$ ($i = 0, 1, \ldots, m-2$). Now by using these numerical values at sample points and the expression of $\widetilde{\Phi}_{k+1}$ in terms of $\{c_i\}_{i=0}^{m-2}$, we deduce

$$\boldsymbol{A}\begin{pmatrix} c_{m-2} \\ c_{m-3} \\ \vdots \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} \widetilde{\Psi}_{k+1}(\omega_0) \\ \widetilde{\Psi}_{k+1}(\omega_1) \\ \vdots \\ \widetilde{\Psi}_{k+1}(\omega_{m-3}) \\ \widetilde{\Psi}_{k+1}(\omega_{m-2}) \end{pmatrix}, \quad (14)$$

where

$$\boldsymbol{A} = \begin{pmatrix} \cos\frac{m-2}{2}\omega_0 & \sin\frac{m-2}{2}\omega_0 & \cdots & \cdots \\ \cos\frac{m-2}{2}\omega_1 & \sin\frac{m-2}{2}\omega_1 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cos\frac{m-2}{2}\omega_{m-3} & \sin\frac{m-2}{2}\omega_{m-3} & \cdots & \cdots \\ \cos\frac{m-2}{2}\omega_{m-2} & \sin\frac{m-2}{2}\omega_{m-2} & \cdots & \cdots \end{pmatrix}. \quad (15)$$

To determine $\{c_i\}_{i=0}^{m-2}$ uniquely, we propose the following careful selection of sample points ω_i ($i = 0, 1, \ldots, m-2$).

[Theorem 1] *The matrix* $\boldsymbol{A}$ *in* (15) *is invertible, hence* $\{c_i\}_{i=0}^{m-2}$ *is determined uniquely by* (14) *if we chose* ω_i *as follows.*

(a) *If* m *is even, i.e.,* $m - 2 = 2l$, *for some* $l \in \mathbb{N}$, $\omega_0 = 0$. *For* $1 \leq i \leq l$, $\omega_i \in (0, \pi)$, $\omega_i \neq \omega_j$ $(i \neq j)$ *and* $\omega_{l+i} = -\omega_i$.

(b) *If* m *is odd, i.e.,* $m - 2 = 2l + 1$, *for some* $l \in \mathbb{N}$, $\omega_0 = 0$, $\omega_{2l+1} = \pi$. *For* $1 \leq i \leq l$, $\omega_i \in (0, \pi)$, $\omega_i \neq \omega_j$ $(i \neq j)$ *and* $\omega_{l+i} = -\omega_i$.

By Theorem 1, we can obtain $\widetilde{\Phi}_{k+1}(\omega)$ which satisfies at least $\widetilde{\Psi}_{k+1}(\omega_i) = \widetilde{\Phi}_{k+1}(\omega_i)$ ($i = 0, 1, \ldots, m-2$) within finite precision of digital computer. Finally, we can construct $\{\widetilde{\Phi}_k(\omega)\}_{k=0}^q$ which satisfies (11). We name this numerical stabilization technique *the mixed trigonometric interpolation*.

4. Improvement of Mixed Trigonometric Interpolation with FFT

The mixed trigonometric interpolation in Sect. 3 has two weak points. First, the computational complexity for solving the system of linear equations (14) is in general $O(n^3)$. Therefore, if the degree of polynomial is very large, it takes a little computational time to obtain $\{\widetilde{\Phi}_k(\omega)\}_{k=0}^q$. Second, if the degree of polynomial is very large, the matrix $\boldsymbol{A}$ in (15) tends to be ill-conditioned.

In this section, we present a technique which not only accelerates but also stabilizes *the mixed trigonometric interpolation*.

Consider to determine the coefficients $\{c_s\}_{s=0}^m$ of a self-reciprocal polynomial

$$\widetilde{D}_k(z) = \sum_{s=0}^{m} c_s z^s \quad \text{(s.t. } c_m c_0 \neq 0)$$

from $\{\widetilde{D}_k(z_n)\}_{n=0}^{N-1}$ for some $N \geq m+1$, where $\widetilde{D}_{kF}^{\dagger}(\omega) = \widetilde{\Phi}_k(\omega)$. (Note: We can determine immediately the coefficients of $\widetilde{\Phi}_k(\omega)$ from $\{c_s\}_{s=0}^m$ by (4)). If we use $N := 2^t$ (s.t. $t \in \mathbb{N}$ and $2^{t-1} < m+1 \leq 2^t$) and $z_n := e^{j\frac{2n\pi}{N}}$ ($n = 0, 1, \ldots, N-1$), we have the solution of (14) by

$$c_s = \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\frac{2n(s-\mathrm{c\,deg}(\widetilde{D}_k))\pi}{N}}\widetilde{\Phi}_k\left(\frac{2n\pi}{N}\right) \quad (s = 0, 1, \ldots, m), \quad (16)$$

which requires only $O(N\log_2 N)$ with FFT.

5. Numerical Examples

5.1 Stabilization by the proposed techniques

In this section, we examine the performance of the proposed method in the algebraic phase unwrapping, along the unit circle, of a univariate complex polynomial which satisfies (1). We define the following criterion to evaluate the gap between $\widetilde{\Psi}_k(\omega)$ and $\widetilde{\Phi}_k(\omega)$ (see (12) and (13))

$$\sigma_{error}(\omega) := \sum_{k=2}^{q}\left|\frac{\widetilde{\Psi}_k(\omega) - \widetilde{\Phi}_k(\omega)}{\widetilde{\Psi}_k(\omega)}\right|. \quad (17)$$

[Example 2] *We define a self-reciprocal polynomial $A_{(0)}(z)$ of degree 69 whose all coefficients' are generated by the uniform distribution for $[-10000, 10000] + j[-10000, 10000]$ and define a self-reciprocal polynomial $A_{(1)}(z)$ of degree 43 whose all coefficients' are generated by the uniform distribution for $[-10000, 10000] + j[-10000, 10000]$. The estimations of the unwrapped phase $\theta_A(\omega)$ for $A(z) := A_{(0)}z + jA_{(1)}z$ with (i) direct SGA, (ii) mixed trigonometric interpolation with Theorem 1 and (iii) mixed trigonometric interpolation with FFT are depicted in Fig. 3 and the gaps between $\widetilde{\Psi}_k(\omega)$ and $\widetilde{\Phi}_k(\omega)$ with each techniques are depicted in Fig. 4. From Fig. 3, the direct SGA fails in exact phase unwrapping at 0.04π and 1.92π. The mixed trigonometric interpolation with Theorem 1 successes in exact phase unwrapping at 1.92π but fails at 0.04π. The mixed trigonometric interpolation with FFT successes in exact phase unwrapping at all $\omega \in (0, 2\pi]$. From Fig. 4, the mixed trigonometric interpolation with FFT achieves the smallest gap between $\widetilde{\Psi}_k(\omega)$ and $\widetilde{\Phi}_k(\omega)$ in average. The direct SGA results in the largest gap is in average.*

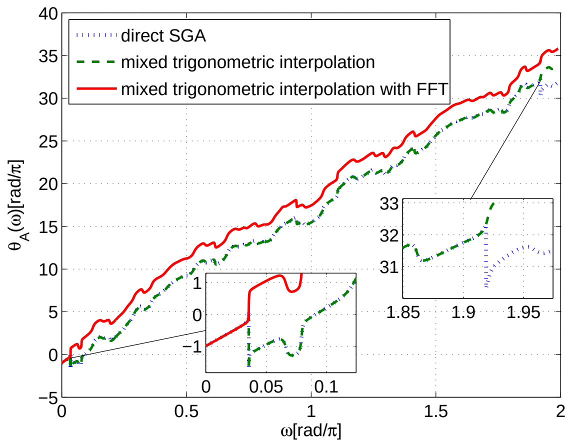

Fig. 3 The estimations of the unwrapped phase $\theta_A(\omega)$ by direct SGA and by stabilization techniques (Example 2)

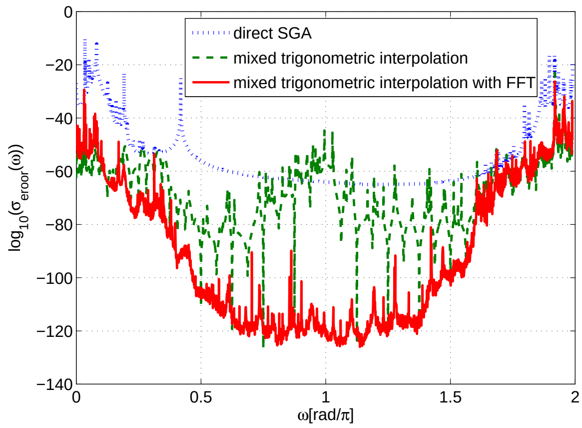

Fig. 4 The gap between $\widetilde{\Psi}_k(\omega)$ and $\widetilde{\Phi}_k(\omega)$ (Example 2)

5.2 Computational time of the proposed techniques

[Example 3] *For a polynomial $A(z)$ of $\deg(A) = 1 \sim 300$ whose all coefficients' are generated by the uniform distribution for $[0, 10] + j[0, 10]$, Fig. 5 shows the computational time for obtaining $\{\widetilde{\Phi}_k(\omega)\}_{k=0}^{\deg(A)+1}$. Solid line expresses the computational time for solving the system of linear equations (14) with Theorem 1 and dashed line expresses the computational time by (16) with FFT. From Fig. 5, we verified that FFT helps greatly to accelerate computation of $\{\widetilde{\Phi}_k(\omega)\}_{k=0}^{n+1}$ especially for polynomials of large degree.*

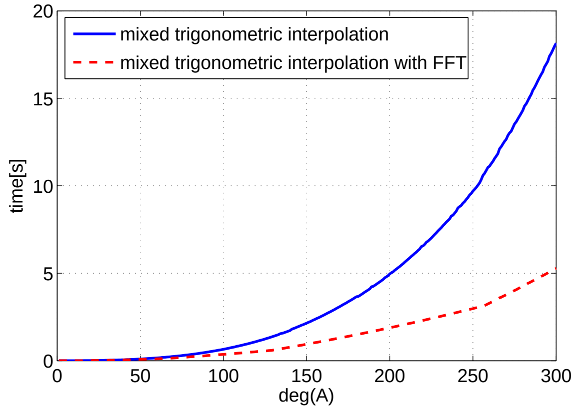

Fig. 5 Computational time for obtaining $\{\widetilde{\Phi}_k(\omega)\}_{k=0}^{\deg(A)+1}$ (Example 3)

6. Conclusion

A pair of numerical stabilization techniques named *the mixed trigonometric interpolation* are presented for the algebraic phase unwrapping. By combining *the mixed trigonometric interpolation* with FFT, we succeeded in making the algebraic phase unwrapping faster and more stable. Numerical examples demonstrate the effectiveness of the proposed techniques.

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