A Stable Sign Estimation for Algebraic Phase Unwrapping

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Abstract In this paper, we propose a simple but a powerful numerical stabilization technique named *the mixed trigonometric interpolation* for the algebraic phase unwrapping. The proposed stabilization technique produces a good approximation of the ideal *general Sturm sequence* by reducing the inductive step in (SGA) to a certain system of linear equations. Thanks to the key property guaranteed by the approximation by the proposed stabilization technique, the algebraic phase unwrapping (SGA) is stabilized greatly even for polynomials of large degrees, hence applicable to practical array signal processing problems formulated with large polynomials.

1 Introduction

A rigorous symbolic algebraic solution to the phase unwrapping problem was established in 1998 [1], where for a given complex polynomial $A(z) \in \mathbb{C}[z]$ satisfying

$$\left.\begin{array}{ll} A(z) \neq 0 & \text{for all } |z| = 1 \\ \Re\{A(1)\} \neq 0 & \end{array}\right\}. \qquad (1)$$

With the algorithm in [1] (See Proposition 1 and Fig. 1 in Sec. 2.2), we can compute for $\omega^* \in (0, 2\pi]$

$$\theta_A(\omega^*) = \theta_A(0) + \int_0^{\omega^*} \left(\arctan \left\{ \frac{\Im\{A(e^{j\omega})\}}{\Re\{A(e^{j\omega})\}} \right\} \right)' d\omega,$$

where $-\pi < \theta_A(0) \leq \pi$ and $A(1) = |A(1)|e^{j\theta_A(0)}$, without using any numerical root finding or numerical integration technique.

Recently, it was shown in [2] that the algebraic phase unwrapping can be applied as a powerful mathematical tool to compute the *Minimum-Maximum distributions of self-reciprocal Laurent polynomial along the unit circle*, which implies that the algorithm can be applied to the estimation of *the Directions-of-Arrival distribution* (DOA distribution): the number of directions of signals in an arbitrarily specified range, which is a valuable information in many array signal processing applications [3] in analogy with the idea of the MUSIC algorithm [4].

However, in a direct computer implementation of the algebraic phase unwrapping algorithm (SGA) in Fig. 1 for polynomials of large degrees, we encounter certain serious instabilities due to the unavoidable gap between numerical value computed by digital computer and theoretical value. Therefore, thoughtless direct computation of (SGA) for polynomials of large degrees, often results in the failure of a key property of the desired *general Sturm sequence*, which is generated by applying (SGA), leading thus the failure of the exact phase unwrapping in the end.

In this paper, we propose a simple but a powerful numerical stabilization technique named *the mixed trigonometric interpolation* for the algebraic phase unwrapping.

2 Preliminaries

2.1 Notation

Let $\mathbb{C}$ denote the set of all complex numbers. We use $j \in \mathbb{C}$ to denote the imaginary unit satisfying $j^2 = -1$. For any $c \in \mathbb{C}$, $\Re(c)$, $\Im(c)$ and $\bar{c}$ stand respectively for the real part, the imaginary part, and complex conjugate of c. For any $C(z) = \sum_{k=l}^{m} c_k z^k \in \mathbb{C}[z, z^{-1}]$ (s.t. $c_l c_m \neq 0$ and $l \leq m$), we define $\deg(C) := m$, $\mathrm{l}\deg(C) := l$, $\mathrm{c}\deg(C) := \frac{l+m}{2}$, $C^*(z) := \sum_{k=l}^{m} \bar{c}_{m-k+l} z^k$, $C_F(\omega) := C(e^{j\omega})$, $C^\dagger(z) := z^{-\mathrm{c}\deg(C)} C(z) \in \mathbb{C}[z^{1/2}, z^{-1/2}]$, $C_F^\dagger(\omega) := C^\dagger(e^{j\omega})$, $C_{(0)}(z) := \frac{C(z)+C^*(z)}{2}$ and $C_{(1)}(z) := \frac{C(z)-C^*(z)}{2j}$. (Note: The degrees of the constant 0 are defined as $\deg(0) = \mathrm{l}\deg(0) = \mathrm{c}\deg(0) = 0$). In particular, $C(z) \in \mathbb{C}[z, z^{-1}]$ satisfying $C(z) = C^*(z)$ is called a self-reciprocal Laurent polynomial. If $C(z)$ is self-reciprocal , $C_F^\dagger(\omega)$ is a real-valued differentiable function over $\mathbb{R}$. For any $C(z) \in \mathbb{C}[z, z^{-1}]$, we have $C(z) = C_{(0)}(z) + jC_{(1)}(z)$, where $C_{(0)}(z)$ and $C_{(1)}(z)$ are self-reciprocal, and $\mathrm{c}\deg(C) = \mathrm{c}\deg(C_{(0)}) = \mathrm{c}\deg(C_{(1)})$. Moreover, we have

$$\Re\{C_F^\dagger(\omega)\} = C_{(0)F}^\dagger(\omega),\ \Im\{C_F^\dagger(\omega)\} = C_{(1)F}^\dagger(\omega). \quad (2)$$

2.2 Algebraic phase unwrapping

Proposition 1 *(Algebraic phase unwrapping along the unit circle [1]) Suppose that $A(z) \in \mathbb{C}[z]$ satisfying (1) and $\theta_A(\omega^*)$ denotes the unwrapped phase of $A_F(\omega)$ at*

$\omega^* \in [0, 2\pi]$. *Define*

$$\mathcal{Z}^{\dagger}_{A_{(0)}} := \{\omega \in [0,2\pi] \mid A^{\dagger}_{(0)F}(\omega) = 0\} = \begin{cases} \emptyset & \text{if } A^{\dagger}_{(0)F}(\omega) \neq 0 \\ & \text{for all } \omega \in [0,2\pi], \\ \{\nu_1, \nu_2, \ldots, \nu_r\} & \text{otherwise}, \end{cases} \quad (3)$$

where $\nu_0(:=)0 < \nu_1 < \cdots < \nu_r < 2\pi$, *and*

$$\mathcal{X}(\nu_i) := \begin{cases} +1 \text{ if} \begin{cases} \mathcal{F}(\omega) > 0 & \text{for } \omega \in (\nu_i - \varepsilon, \nu_i) \text{ and} \\ \mathcal{F}(\omega) < 0 & \text{for } \omega \in (\nu_i, \nu_i + \varepsilon), \end{cases} \\ -1 \text{ if} \begin{cases} \mathcal{F}(\omega) < 0 & \text{for } \omega \in (\nu_i - \varepsilon, \nu_i) \text{ and} \\ \mathcal{F}(\omega) > 0 & \text{for } \omega \in (\nu_i, \nu_i + \varepsilon), \end{cases} \\ 0 \quad \text{otherwise}, \end{cases} \quad (4)$$

for ν_i $(i = 1, 2, \ldots, r)$ *and for sufficiently small* $\varepsilon > 0$, *where* $\mathcal{F}(\omega) := A^{\dagger}_{(0)F}(\omega) A^{\dagger}_{(1)F}(\omega)$. *Then we have the following relations.*

(a) *For any* $\omega^* \in (0, 2\pi]$,

$$\begin{aligned} \theta_A(\omega^*) &= \theta_A(0) + \int_0^{\omega^*} \left(\arctan \left\{ \frac{\Im\{A_F(\omega)\}}{\Re\{A_F(\omega)\}} \right\} \right)' d\omega \\ &= \theta_A(0) + \mathrm{c\,deg}(A)\omega^* - \arctan\{Q^{\dagger}_A(0)\} \\ &\quad + \lim_{\nu \to \omega^* - 0} \arctan\{Q^{\dagger}_A(\nu)\} + \Lambda(\omega^*)\pi, \end{aligned} \quad (5)$$

where $Q^{\dagger}_A(\omega) := \frac{A^{\dagger}_{(1)F}(\omega)}{A^{\dagger}_{(0)F}(\omega)}$ *and* $\Lambda(\omega^*) := \sum_{\nu_i \in (0, \omega^*)} \mathcal{X}(\nu_i)$.

(b) *Let* $\{\Phi_k(\omega)\}_{k=0}^{q}$ *be a sequence of functions over* $0 \leq \omega \leq 2\pi$ *obtained by applying the algorithm (SGA) in Fig. 1 to* $A_{(0)}(z)$ *and* $A_{(1)}(z)$ *with*

$$\left.\begin{aligned} D_k(z) &:= z^{-1\deg(A_{(k)})} \left(\tfrac{j}{z-1}\right)^{e_k} A_{(k)}(z) \\ \Phi_k(\omega) &:= D^{\dagger}_{kF}(\omega) \quad \text{for } k = 0, 1, 2, \ldots, q \end{aligned}\right\}, \quad (6)$$

where e_k *denotes the order of* $z = 1$ *as a zero of polynomial* $A_k(z)$. *Define for each* $\omega \in [0, 2\pi]$ *the number of variations in the sign of* $\{\Phi_k(\omega)\}_{k=0}^{q}$ *by*

$$\begin{aligned} V\{\Phi(\omega)\} &:= V\{\Phi_0(\omega), \Phi_1(\omega), \ldots, \Phi_q(\omega)\} \\ &= \left|\{i \mid 0 \leq i < q \text{ and } \Phi_i(\omega)\Phi_{i+\mathcal{Q}(i)}(\omega) < 0\}\right|, \end{aligned} \quad (7)$$

where $\mathcal{Q}(i) := \min\{k \in \mathbb{N} \mid \Phi_{i+k}(\omega) \neq 0\}$ *(i.e.,* $V\{\Phi(\omega)\}$ *denotes the number of sign changes of the entries in* $\{\Phi_k(\omega)\}_{k=0}^{q}$ *when these are scanned sequentially from left to right. If there exists some* Φ_k *whose value at* ω *is* $\Phi_k(\omega) = 0$, *its sign is not counted). Then, for every* $\omega^* \in (0, 2\pi]$, *we have*

$$\begin{aligned} \theta_A(\omega^*) &= \theta_A(0) + \mathrm{c\,deg}(A)\omega^* - \arctan\{Q^{\dagger}_A(0)\} \\ &\quad + \begin{cases} \arctan\{Q^{\dagger}_A(\omega^*)\} + [V\{\Phi(\omega^*)\} - V\{\Phi(0)\}]\pi \\ \qquad \text{if } A^{\dagger}_{(0)F}(\omega^*) \neq 0, \\ \pi/2 + [V\{\Phi(\omega^*)\} - V\{\Phi(0)\}]\pi \\ \qquad \text{if } A^{\dagger}_{(0)F}(\omega^*) = 0. \end{cases} \end{aligned} \quad (8)$$

begin
  Define $A_{(0)}(z)$ and $A_{(1)}(z)$.
  if $\deg(D_1) = 0$,
  then $p := 1$
  else
  begin
    $k := 1$;
    Repeat
      $v_k := \deg(D_{k-1}) - \deg(D_k)$
      $\beta_k := \frac{D_{k-1}(0)}{D_k(0)}$, $\gamma_k := j^{1-v_k}\beta_k$
      $A_{(k+1)}(z) := \begin{cases} (\beta_k + \bar{\beta}_k z^{v_k})D_k(z) - D_{k-1}(z) & \text{if } v_k > 0, \\ (\gamma_k + \bar{\gamma}_k z)D_k(z) - (\frac{z-1}{j})^{1-v_k} D_{k-1}(z) & \text{if } v_k \leq 0, \end{cases}$
      $k := k + 1$
    Until $\deg(D_k) = 0$ $(k \geq 2)$
    $p := k$
  end
  $q := \begin{cases} p & \text{if } D_p(z) \not\equiv 0 \\ p - 1 & \text{if } D_p(z) \equiv 0 \end{cases}$
end;

Figure 1: Algorithm to generate general Sturm sequence (SGA)

In this paper, we call the sequence of functions $\{\Phi_k(\omega)\}_{k=0}^{q}$ *general Sturm sequence*.

Example 1 *By using the SGA and Proposition 1, let us construct the unwrapped phase of the univariate complex polynomial*

$$A(z) := -(2-8j)z^3 + (6-4j)z^2 - (4-6j)z + (6-10j),$$

which satisfies (1). Then, the result is the follows and Fig. 2.

$$\begin{aligned} A_{(0)}(z) &= (2+9j)z^3 + (1-5j)z^2 + (1+5j)z + (2-9j), \\ A_{(1)}(z) &= -(1-4j)z^3 + (1-5j)z^2 + (1+5j)z - (1+4j), \\ D_0(z) &= (2+9j)z^3 + (1-5j)z^2 + (1+5j)z + (2-9j), \\ D_1(z) &= -(4+j)z^2 + z - (4-j), \\ D_2(z) &= (4-4j)z + (4+4j), \\ D_3(z) &= 1, \\ \Phi_0(\omega) &= 4\cos\frac{3}{2}\omega - 18\sin\frac{3}{2}\omega + 2\cos\frac{\omega}{2} + 10\sin\frac{\omega}{2}, \\ \Phi_1(\omega) &= -8\cos\omega + 2\sin\omega + 1, \\ \Phi_2(\omega) &= 8\cos\frac{\omega}{2} + 8\sin\frac{\omega}{2}, \\ \Phi_3(\omega) &= 1. \end{aligned}$$

2.3 Instabilities in (SGA)

To implement the algorithm (SGA) in Fig. 1 precisely, we need large number of digits to express the rational coefficients of the polynomials $A_{(k)}(z)$ or $D_k(z)$

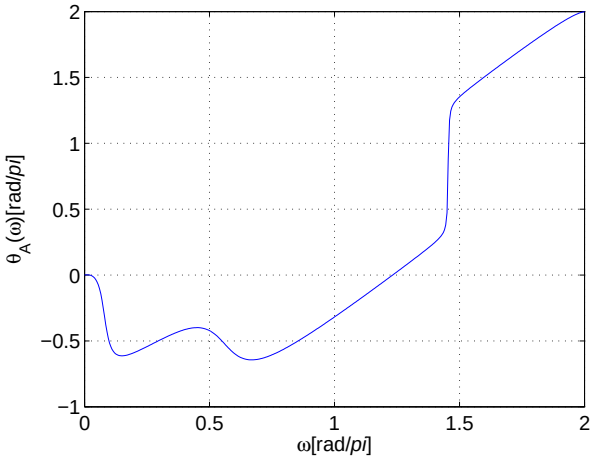

Figure 2: Exact unwrapped phase by Proposition 1 (Example 1)

($k = 0, 1, \ldots, q$) mainly due to the repeated computations of β_k ($k = 1, 2, \ldots, q-1$). We call this phenomenon the *coefficient growth* in analogy with the typical cases in the computation of the *standard Sturm sequence* through the Euclid's algorithm [5]. In computer implementation of (SGA), we encounter certain serious instabilities due to, e.g., (i) the truncation error of the trigonometric function values, and (ii) the coefficient growth which causes the truncation error in the floating-point expression of the rational coefficients (or memory shortages by increasing number of digits for exact expression of the rational coefficients). Thoughtless direct computation of (SGA) often results in the failure of the key property:

$$\Phi_k(\omega_0) = 0 \text{ at } \omega_0 \in [0, 2\pi] \Rightarrow \Phi_{k-1}(\omega_0)\Phi_{k+1}(\omega_0) < 0 \quad \text{for } 0 < k < q(\geq 2), \tag{9}$$

leading thus the failure of (8). This situation restricts the practical applicability of Proposition 1 especially for polynomials $A(z) \in \mathbb{C}[z]$ of large degrees.

In the next section, we propose a simple but powerful stabilization technique to maintain the key property (9).

3 Numerical stabilization of algebraic phase unwrapping

For numerical implementation of the algebraic phase unwrapping through (SGA), we need a certain stable approximation of the value $V\{\Phi(\omega)\}$. This goal is achieved by a certain numerical approximation $\{\widetilde{\Phi}_k(\omega)\}_{k=0}^q (\approx \{\Phi_k(\omega)\}_{k=0}^q)$ which is guaranteed to satisfy

$$\widetilde{\Phi}_0 = \Phi_0, \widetilde{\Phi}_1 = \Phi_1 \text{ and for } 0 < k < q(\geq 2),$$
$$\widetilde{\Phi}_k(\omega_0) = 0 \text{ at } \omega_0 \in [0, 2\pi] \Rightarrow \widetilde{\Phi}_{k-1}(\omega_0)\widetilde{\Phi}_{k+1}(\omega_0) < 0. \tag{10}$$

Indeed, by (10), we can obtain $V\{\widetilde{\Phi}(\omega)\} (= V\{\Phi(\omega)\})$ unless ω is in the vicinity of zeros of Φ_0. (Note: The unavoidable gap between numerical and theoretical zeros of Φ_0 does not guarantee $V\{\widetilde{\Phi}(\omega)\} = V\{\Phi(\omega)\}$ for ω in the vicinity of such zeros). The proposed construction of $\{\widetilde{\Phi}_k(\omega)\}_{k=0}^q (\approx \{\Phi_k(\omega)\}_{k=0}^q)$ is presented inductively as follows.

Suppose that we have

$$\left.\begin{aligned} \widetilde{\Phi}_{k-1}(\omega) &= a_m \cos\tfrac{m}{2}\omega + a_{m-1}\sin\tfrac{m}{2}\omega + a_{m-2}\cos\tfrac{m-2}{2}\omega + \cdots \\ \widetilde{\Phi}_k(\omega) &= b_n \cos\tfrac{n}{2}\omega + b_{n-1}\sin\tfrac{n}{2}\omega + b_{n-2}\cos\tfrac{n-2}{2}\omega + \cdots \end{aligned}\right\}.$$

Under the standard assumption[1], we can express $\widetilde{\Phi}_{k+1}(\omega)$ for $m > n$ by

$$\widetilde{\Phi}_{k+1}(\omega) = c_{m-2}\cos\tfrac{m-2}{2}\omega + c_{m-3}\sin\tfrac{m-2}{2}\omega + c_{m-4}\cos\tfrac{m-4}{2}\omega + \cdots,$$

for $m \leq n$ by

$$\widetilde{\Phi}_{k+1}(\omega) = c_{n-1}\cos\tfrac{n-1}{2}\omega + c_{n-2}\sin\tfrac{n-1}{2}\omega + c_{n-3}\cos\tfrac{n-3}{2}\omega + \cdots.$$

In the following construction of $\widetilde{\Phi}_{k+1}(\omega)$, we assume for simplicity, the case $m > n$ (Note: The discussion for the other case is almost same). To determine $\{c_k\}_{k=0}^{m-2}$ in order to satisfy (10) without suffering the coefficient growth in the direct computation of (SGA), we reduce this problem to an interpolation problem of the mixed trigonometric function $\widetilde{\Phi}_{k+1}(\omega)$. By choosing carefully sample points $\omega_i \in [0, 2\pi]$ ($i = 0, 1 \ldots, m-2$) for this interpolation problem, we can determine $\{c_k\}_{k=0}^{m-2}$ uniquely by solving a system of linear equations.

Applying (SGA) directly to $\widetilde{\Phi}_{k-1}(\omega)$ and $\widetilde{\Phi}_k(\omega)$, we obtain an expression of the desired $\widetilde{\Phi}_{k+1}(\omega)$ as

$$\widetilde{\Phi}_{k+1}(\omega) = \frac{2\widetilde{\Phi}_k(\omega)}{b_n^2 + b_{n-1}^2}\left\{(a_m b_n + a_{m-1}b_{n-1})\cos\frac{m-n}{2}\omega - (a_m b_{n-1} - a_{m-1}b_n)\sin\frac{m-n}{2}\omega\right\} - \widetilde{\Phi}_{k-1}(\omega). \tag{11}$$

Unfortunately, thoughtless direct computation of $\{c_k\}_{k=0}^{m-2}$ through the elementary trigonometric expansion causes the coefficient growth as remarked in Sec. 2.3.

We take a different path to find a stable numerical approximation of $\{c_k\}_{k=0}^{m-2}$ by using the relation (11) for numerical evaluation of $\widetilde{\Phi}_{k+1}(\omega_i)$ ($i = 0, \ldots, m-2$). Now by using these numerical values at sample points and the expression of $\widetilde{\Phi}_{k+1}$ in terms of $\{c_k\}_{k=0}^{m-2}$, we

[1] Almost always, we can assume (i) $A_{(k)}(1) \neq 0 (k = 2, 3, \ldots, q)$ and (ii) $\left.\begin{matrix}\deg(D_{k-1}) = m \\ \deg(D_k) = n\end{matrix}\right\} \Rightarrow \deg(D_{k+1}) = \begin{cases} m-2 & \text{if } m > n \\ n-1 & \text{if } m \leq n \end{cases}$

deduce

$$\boldsymbol{A}\begin{pmatrix} c_{m-2} \\ c_{m-3} \\ \vdots \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} \widetilde{\Phi}_{k+1}(\omega_0) \\ \widetilde{\Phi}_{k+1}(\omega_1) \\ \vdots \\ \widetilde{\Phi}_{k+1}(\omega_{m-3}) \\ \widetilde{\Phi}_{k+1}(\omega_{m-2}) \end{pmatrix}, \qquad (12)$$

where

$$\boldsymbol{A} = \begin{pmatrix} \cos\frac{m-2}{2}\omega_0 & \sin\frac{m-2}{2}\omega_0 & \cdots & \cdots \\ \cos\frac{m-2}{2}\omega_1 & \sin\frac{m-2}{2}\omega_1 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cos\frac{m-2}{2}\omega_{m-3} & \sin\frac{m-2}{2}\omega_{m-3} & \cdots & \cdots \\ \cos\frac{m-2}{2}\omega_{m-2} & \sin\frac{m-2}{2}\omega_{m-2} & \cdots & \cdots \end{pmatrix}. \qquad (13)$$

To determine $\{c_k\}_{k=0}^{m-2}$ uniquely, we propose the following careful selection of sample points ω_i ($i = 0, \ldots, m-2$).

Theorem 1 *The matrix $\boldsymbol{A}$ in (13) is invertible hence $\{c_k\}_{k=0}^{m-2}$ is determined uniquely by (12) if we chose ω_i as follows.*

(a) *When m is even, i.e., $m-2 = 2k$, for some $k \in \mathbb{N}$, $\omega_0 = 0$. For $1 \le i \le k$, $\omega_i \in (0, \pi)$, $\omega_i \neq \omega_j$ $(i \neq j)$ and $\omega_{k+i} = -\omega_i$.*

(b) *When m is odd, i.e., $m-2 = 2k+1$, for some $k \in \mathbb{N}$, $\omega_0 = 0$, $\omega_{2k+1} = \pi$. For $1 \le i \le k$, $\omega_i \in (0, \pi)$, $\omega_i \neq \omega_j$ $(i \neq j)$ and $\omega_{k+i} = -\omega_i$.*

We name the above numerical stabilization technique (based on Theorem 1) *the mixed trigonometric interpolation*.

To obtain the value of $V\{\widetilde{\Phi}(\omega)\}$ as precise as possible, we additionally introduce a simple multiplication of the coefficients $\widetilde{\Phi}_k$ by $2^{\pm 1}$. If there is a coefficient of $\widetilde{\Phi}_k$ whose absolute value is too small or too large to evaluate numerically the value of $\widetilde{\Phi}_{k+1}(\omega)$, all coefficients are multiplied by 2 or $1/2$ repeatedly until the absolute values of all coefficients fall between the lower and upper limits. (Note: Multiplication by 2 or $1/2$ does not cause any numerical error in digital computer).

4 Numerical example

We apply the numerical stabilization techniques in Sec. 3 to the phase unwrapping of

$$\begin{aligned} A(z) := &-(120294+12204j)z^{11}+(16155+16551j)z^{10} \\ &+14145jz^9+(11513+15115j)z^8-(15512+20397j)z^7 \\ &+(19512-24151j)z^6-(21515-21515j)z^4+12431z^3 \\ &-10243jz^2+(173316+16362j)z-16693j, \end{aligned}$$

which satisfies (1). This is an example for which a direct computation of (SGA) fails to return the exact unwrapped phase due to the severe coefficient growth remarked in Sec. 2.3. From the result shown in Fig. 3, we verified that the exact phase unwrapping is achieved successfully by the proposed stabilization techniques.

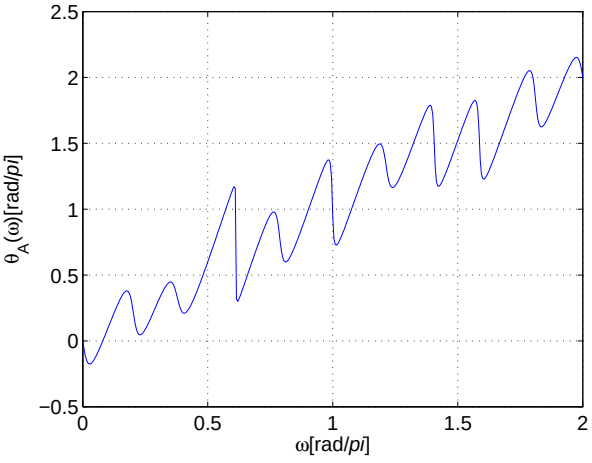

Figure 3: Exact unwrapped phase with *the mixed trigonometric interpolation*

5 Conclusion

A simple but powerful numerical stabilization technique named *the mixed trigonometric interpolation* is presented for the algebraic phase unwrapping problem. Application of the proposed technique to an example, which causes the severe coefficient growth by a direct computation of (SGA), demonstrates the effectiveness of the proposed technique.

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