A Stabilization of Algebraic Phase Unwrapping along the Unit Circle with Self-Reciprocal Subresultant

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1 Algebraic Phase Unwrapping along the Unit Circle

Algebraic phase unwrapping along the unit circle [1] gives the exact closed-form expression of the unwrapped phase of a complex polynomial along the unit circle for, e.g., evaluation of the stability of a certain digital filter and computation of the *complex cepstrum*. Theorem 1 below is given as a refinement of [1, Theorem 6] by redefining the *Sturm sequence* (cf. lines 4, 5, 7 & 8 in Algorithm 1 and [1, (33) & (35)]).

Theorem 1 *Let* $A(z) \in \mathbb{C}[z]$ *satisfy* $A_F(\omega) := A(e^{\imath\omega}) \neq 0$ *for all* $\omega \in [0, 2\pi]$, $A_{(0)} := \frac{A+A^*}{2} \neq 0$ *and* $A_{(1)} := \frac{A-A^*}{2\imath} \neq 0$.[1] *Let* $(\Phi_k(\omega))_{k=0}^{q}$ *be the Sturm sequence generated by SGA.*[2] *For all* $\omega^* \in (0, 2\pi]$, *the unwrapped phase of A along the unit circle (or the unwrapped phase of* A_F*) can be computed from*

$$\theta_{A_F}(\omega^*) - \theta_{A_F}(0) := \int_0^{\omega^*} \Im\left[\frac{A_F'(\omega)}{A_F(\omega)}\right] \mathrm{d}\omega$$

$$= \mathrm{cdeg}(A)\omega^* - \begin{cases} \arctan(\mathcal{Q}_A^\dagger(0)) & \text{if } A_{(0)F}^\dagger(0) \neq 0; \\ \mathrm{sgn}(\Phi_0(0)\Phi_1(0))\pi/2 & \text{if } A_{(0)F}^\dagger(0) = 0; \end{cases}$$

$$+ \begin{cases} \arctan(\mathcal{Q}_A^\dagger(\omega^*)) + [V(\boldsymbol{\Phi}(\omega^*)) - V(\boldsymbol{\Phi}(0))]\pi \\ \qquad\qquad\qquad\qquad \text{if } A_{(0)F}^\dagger(\omega^*) \neq 0; \\ \pi/2 + [V(\boldsymbol{\Phi}(\omega^*)) - V(\boldsymbol{\Phi}(0))]\pi \quad \text{if } A_{(0)F}^\dagger(\omega^*) = 0, \end{cases} \tag{1}$$

where $\mathcal{Q}_A^\dagger(\omega) := A_{(1)F}^\dagger(\omega)/A_{(0)F}^\dagger(\omega)$, $\mathrm{sgn}(x) := x/|x|$ *if* $x \neq 0$, $\mathrm{sgn}(x) := 0$ *if* $x = 0$, *and* $V : \mathbb{R}^{q+1} \to \mathbb{Z}_+$ *counts the number of sign changes in* $(\Phi_0(\omega^*), \Phi_1(\omega^*), \ldots, \Phi_q(\omega^*)) =: \boldsymbol{\Phi}(\omega^*)$.

Algorithm 1 Sturm Generating Algorithm (SGA)

Input: $A_{(0)}(z) = A_{(0)}^*(z) \in \mathbb{C}[z]$ and $A_{(1)}(z) = A_{(1)}^*(z) \in \mathbb{C}[z]$

Output: $(\Phi_k(\omega))_{k=0}^{q}$

1: $\widetilde{D}_0(z) \leftarrow z^{-\mathrm{ldeg}(A_{(0)})}(\frac{\imath}{z-1})^{o_0} A_{(0)}(z)$ (o_0: order of $z = 1$ as a zero of $A_{(0)}$)

2: $\widetilde{D}_1(z) \leftarrow z^{-\mathrm{ldeg}(A_{(1)})}(\frac{\imath}{z-1})^{o_1} A_{(1)}(z)$ (o_1: order of $z = 1$ as a zero of $A_{(1)}$)

3: $\Phi_0(\omega) \leftarrow \widetilde{D}_0^\dagger(e^{\imath\omega})$, $\Phi_1(\omega) \leftarrow \widetilde{D}_1^\dagger(e^{\imath\omega})$

4: $(\delta_0, \delta_1) \leftarrow$

$$\begin{cases} (0,0) & \text{if } (\deg(\widetilde{D}_0) + \deg(\widetilde{D}_1)) \text{ is odd} \\ (1,0) & \text{if } (\deg(\widetilde{D}_0) + \deg(\widetilde{D}_1)) \text{ is even and } \deg(\widetilde{D}_0) \geq \deg(\widetilde{D}_1) \\ (0,1) & \text{if } (\deg(\widetilde{D}_0) + \deg(\widetilde{D}_1)) \text{ is even and } \deg(\widetilde{D}_1) > \deg(\widetilde{D}_0) \end{cases}$$

5: $D_0(z) \leftarrow (\frac{z-1}{\imath})^{\delta_0} \widetilde{D}_0(z)$, $D_1(z) \leftarrow (\frac{z-1}{\imath})^{\delta_1} \widetilde{D}_1(z)$, $k \leftarrow 1$

6: **while** $\deg(D_k) \geq 1$ **do**

7: $\quad \widetilde{D}_{k+1}(z) \leftarrow -D_{k-1}(z) - H_k(z)D_k(z) \quad (H_k(z) = H_k^*(z), \widetilde{D}_{k+1} \neq 0 \Rightarrow \mathrm{cdeg}(\widetilde{D}_{k+1}) = \mathrm{cdeg}(D_{k-1})$ and $\deg(\widetilde{D}_{k+1}) - \mathrm{ldeg}(\widetilde{D}_{k+1}) < \deg(D_k))$

8: $\quad \Phi_{k+1}(\omega) \leftarrow \widetilde{D}_{k+1}^\dagger(e^{\imath\omega})$, $D_{k+1}(z) \leftarrow z^{-\mathrm{ldeg}(\widetilde{D}_{k+1})} \widetilde{D}_{k+1}(z)$

9: $\quad k \leftarrow k + 1$

10: **end while**

11: $q \leftarrow \begin{cases} k & \text{if } \Phi_k \neq 0 \\ k-1 & \text{if } \Phi_k = 0 \end{cases}$

[1] If $A_{(0)} = 0$ or $A_{(1)} = 0$, then $\theta_{A_F}(\omega) = \theta_{A_F}(0) + \mathrm{cdeg}(A)\omega$ ($\omega \in [0, 2\pi]$).

[2] In Algorithm 1, for $C(z) = \sum_{k=l}^{m} c_k z^k \in \mathbb{C}[z]$ (s.t. $c_l c_m \neq 0$), $\deg(C) := m$, $\mathrm{ldeg}(C) := l$, $\mathrm{cdeg}(C) := \frac{l+m}{2}$, and $C^*(z) := \sum_{k=l}^{m} \bar{c}_{l+m-k} z^k \in \mathbb{C}[z]$. If $C(z) = C^*(z)$, C is called a *self-reciprocal polynomial*. For $C(z) = C^*(z)$, define $C^\dagger(z) := z^{-\mathrm{cdeg}(C)} C(z)$, then $C_F^\dagger(\omega) := C^\dagger(e^{\imath\omega})$ is real-valued.

2 Stabilization with Self-Reciprocal Subresultant

Unfortunately, a direct computer implementation of Theorem 1 sometimes leads to failure in phase unwrapping due to numerical instabilities caused by *coefficient growth* in the computation of the Sturm sequence. In [2], we presented an alternative idea for stabilizing algebraic phase unwrapping along the real axis with the use of the subresultant [3]. In the following, we propose a similar technique for Theorem 1.

Let $D_0(z) = \sum_{k=0}^{m} a_k z^k = D_0^*(z)$ and $D_1(z) = \sum_{k=0}^{n} b_k z^k = D_1^*(z)$ s.t. $a_m = \bar{a}_0 \neq 0$, $b_n = \bar{b}_0 \neq 0$, $m > n \geq 1$, and $(m+n)$ is odd. We newly define the ith *self-reciprocal subresultant* $\mathrm{SRSres}_i[D_0, D_1](z) \in \mathbb{C}[z^{\frac{1}{2}}, z^{-\frac{1}{2}}]$ ($i = 0, 1, \ldots, n-1$), as the determinant of an $(m+n-2i) \times (m+n-2i)$ matrix, by

$$\mathrm{SRSres}_i[D_0, D_1](z) := \begin{vmatrix}
a_m & a_{m-1} & & \cdots & a_{m-n+i+2} & & a_{m-n+i+1} & a_0 & \cdots & \vdots & a_{\frac{m-n-1}{2}} & D_0^\dagger(z)z^{\frac{n-i-1}{2}} \\
& a_m & & \cdots & a_{m-n+i+3} & a_0 & a_{m-n+i+2} & a_1 & \cdots & \vdots & a_{\frac{m-n+1}{2}} & D_0^\dagger(z)z^{\frac{n-i-3}{2}} \\
& & & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
& & a_0 & \cdots & a_m & a_{n-i-3} & a_{m-1} & a_{n-i-2} & \cdots a_{\frac{m+n-1}{2}} & & \vdots & D_0^\dagger(z)z^{-\frac{n-i-3}{2}} \\
& a_0 & a_1 & \cdots & & a_{n-i-2} & a_m & a_{n-i-2} & \cdots a_{\frac{m+n+1}{2}} & & \vdots & D_0^\dagger(z)z^{-\frac{n-i-1}{2}} \\
b_n & b_{n-1} & & \cdots & b_{i+2} & & b_{i+1} & & \cdots & \vdots & & D_1^\dagger(z)z^{\frac{m-i-1}{2}} \\
& b_n & & \cdots & b_{i+3} & & b_{i+2} & & \cdots & \vdots & & D_1^\dagger(z)z^{\frac{m-i-3}{2}} \\
& & & \ddots & \vdots & & \vdots & & \cdots & \vdots & & \vdots \\
& & & & b_n & & b_{n-1} & & \cdots & \vdots & & \vdots \\
& & & & & & b_n & & \cdots & \vdots & & \vdots \\
& & & & & & & & \cdots & b_{i+2} & b_0 & D_1^\dagger(z)z^{\frac{n-i-2}{2}} \\
& & & & & & & & \cdots & \vdots & \vdots & \vdots \\
& & & & & & & & \cdots & b_n & b_{n-i-2} & D_1^\dagger(z)z^{-\frac{n-i-2}{2}} \\
& & & & & & & b_0 & \cdots & & \vdots & \vdots \\
& & & & & b_0 & & b_1 & \cdots & & \vdots & \vdots \\
& & & \iddots & & \vdots & & \vdots & \cdots & & \vdots & \vdots \\
& & b_0 & \cdots & & b_{n-i-3} & & b_{n-i-2} & \cdots & & \vdots & D_1^\dagger(z)z^{-\frac{m-i-3}{2}} \\
& b_0 & b_1 & \cdots & & b_{n-i-2} & & b_{n-i-1} & \cdots & & \vdots & D_1^\dagger(z)z^{-\frac{m-i-1}{2}}
\end{vmatrix}.$$

Theorem 2 *If* $\deg(\widetilde{D}_0) \geq \deg(\widetilde{D}_1) \geq 1$ *and* $(D_k(z))_{k=0}^{q}$ *is regular, i.e.,* $\deg(D_{k+1}) = \deg(D_k) - 1 = \deg(D_1) - k$ *for* $k = 1, 2, \ldots, q-1$, *then* $q = \deg(D_1) + 1$ *and, for all* $\omega^* \in \mathbb{R}$,

$$\mathrm{sgn}(\Phi_k(\omega^*)) = (-1)^{(k-1)k/2 + (k-2)(\deg(D_0) - \deg(D_1) + k - 2)/2} \cdot \mathrm{sgn}\big(\mathrm{SRSres}_{\deg(D_1)-k+1}[D_0, D_1](e^{\imath\omega^*})\big) \quad (k = 2, 3, \ldots, q). \tag{2}$$

From (2), we can compute the exact signs in $\Phi(\omega^*)$ without computing the coefficients of $\widetilde{D}_k$ ($k = 2, 3, \ldots, q$), and hence we can stably compute $V(\Phi(\omega^*))$ in (1) without suffering from the propagation of errors caused by coefficient growth.

References

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[3] W. S. Brown and J. F. Traub, “On Euclid’s algorithm and the theory of subresultants,” *J. ACM*, vol. 18, no. 4, pp. 505–514, 1971.