Design of Tight Minimum-Sidelobe Windows by Newton's Method on Oblique Manifolds for Time-Frequency Domain Signal Processing

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1 Introduction

The short-time Fourier transform (STFT) or the discrete Gabor transform (DGT) has been widely utilized for signal analysis and processing [1], [2]. STFT/DGT localizes frequency components of a signal at each time with a *window function* whose energy is concentrated in the time-frequency domain. In particular, the *Slepian window* can minimize the *sidelobe energy* [3] in signal analysis.

In STFT/DGT domain signal processing, the result of STFT/DGT is modified and then converted to a time domain signal by the inverse STFT/DGT. According to the frame theory [4], the window is desired to be *tight*, for processing that is robust to noise and has few artifacts. In this paper, we propose to design tight windows minimizing the sidelobe energy. It is expressed as the *maximization of Rayleigh quotients on oblique manifolds*. We apply the *Riemannian Newton's method* [5] to obtain the optimal tight windows by several iterations.

2 STFT/DGT and Tight Windows

Let $\boldsymbol{x} := (x[0], x[1], \ldots, x[L-1])^{\mathrm{T}} \in \mathbb{C}^L$ and $\boldsymbol{w} := (w[0], w[1], \ldots, w[K-1])^{\mathrm{T}} \in \mathbb{R}^K$ be a signal and a window (s.t $K < L$), respectively. Let a and M be integers satisfying $\frac{L}{a} =: N \in \mathbb{N}$ and $0 < a < K \leq M \leq L$. In this paper, we define STFT/DGT of $\boldsymbol{x}$ and its inversion as

$$\begin{cases} X[m,n] = \displaystyle\sum_{l=0}^{K-1} x[l+an]\, w[l]\, \mathrm{e}^{-\frac{2\pi \mathrm{i} m(l+an)}{M}}, \\ x[l] = \displaystyle\sum_{n=\lceil \frac{l-K+1}{a} \rceil}^{\lfloor \frac{l}{a} \rfloor} \gamma[l-an] \sum_{m=0}^{M-1} X[m,n]\, \mathrm{e}^{\frac{2\pi \mathrm{i} ml}{M}}, \end{cases}$$

where $m = 0, 1, \ldots, M-1$, $n = 0, 1, \ldots, N-1$, $\mathrm{i} \in \mathbb{C}$ denotes the imaginary unit, $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are the floor and ceiling functions, the signal $x[\cdot]$ and the coefficients $X[m, \cdot]$ are treated periodically as $x[l+L] := x[l]$ and $X[m,-n] := X[m, N-n]$, and $\boldsymbol{\gamma} := (\gamma[0], \gamma[1], \ldots, \gamma[K-1])^{\mathrm{T}} \in \mathbb{R}^K$ is the *canonical dual window*. Using a diagonal matrix

$$\boldsymbol{S}_{\boldsymbol{w}} := \mathrm{diag}\left(M \sum_{n=-\lfloor \frac{l}{a} \rfloor}^{\lfloor \frac{K-l-1}{a} \rfloor} |w[l+an]|^2 \right)_{l=0}^{K-1},$$

the canonical dual window is given as $\boldsymbol{\gamma} = \boldsymbol{S}_{\boldsymbol{w}}^{-1} \boldsymbol{w}$.

If a window $\boldsymbol{w}$ is self-dual, i.e., $\boldsymbol{\gamma} = \frac{1}{\lambda} \boldsymbol{w}$ holds for some $\lambda > 0$, $\boldsymbol{w}$ is called a *tight window*. When using a tight window, the forward transform from $x[l]$ to $X[m,n]$ is robust to noise, and the inverse transform from modified coefficients $\widetilde{X}[m,n]$ to a processed signal $\widetilde{x}[l]$ is unlikely to create artifacts, according to the frame theory [4]. A tight window $\boldsymbol{w}_{\mathrm{t}}$ can be given from $\boldsymbol{w}$ by the *metric projection*:

$$\boldsymbol{w}_{\mathrm{t}} = \sqrt{\lambda}\, \boldsymbol{S}_{\boldsymbol{w}}^{-1/2} \boldsymbol{w}. \tag{1}$$

3 Tight Minimum-Sidelobe Windows

Define the *discrete-time Fourier transform* of $\boldsymbol{w}$ by $\widehat{w}(f) := \sum_{l=0}^{K-1} w[l]\, \mathrm{e}^{-2\pi \mathrm{i} fl}$ for $f \in [-\frac{1}{2}, \frac{1}{2})$, and set $p \in (0,1)$ as a proportion of the *mainlobe* of $\widehat{w}(f)$. In [3], to minimize the sidelobe energy, the *spectral concentration problem* is considered:

$$\underset{\boldsymbol{w} \in \mathbb{R}^K \setminus \{\boldsymbol{0}\}}{\mathrm{maximize}} \; \frac{\int_{-p/2}^{p/2} |\widehat{w}(f)|^2 \,\mathrm{d}f}{\int_{-1/2}^{1/2} |\widehat{w}(f)|^2 \,\mathrm{d}f} \left(= \frac{\boldsymbol{w}^{\mathrm{T}} \boldsymbol{Q}_p \boldsymbol{w}}{\|\boldsymbol{w}\|_2^2} \right), \tag{2}$$

where $\boldsymbol{Q}_p := (p\,\mathrm{sinc}(p(l-l'))) = (\frac{\sin(\pi p(l-l'))}{\pi(l-l')}) \in \mathbb{R}^{K \times K}$ is a positive-definite symmetric matrix and its *first principal eigenvector* $\boldsymbol{w}_{\mathcal{S},p}$ is the solution and called the Slepian window. $\boldsymbol{w}_{\mathcal{S},p}$ is symmetric and positive, i.e., $w_{\mathcal{S},p}[l] = w_{\mathcal{S},p}[K-l-1] > 0$.

In this paper, we solve the spectral concentration problem in (2) *under the tightness constraint*

$$\sum_{n=0}^{J-1} |w[l+an]|^2 = \frac{\lambda}{M} \quad (l = 0, 1, \ldots, a-1), \tag{3}$$

where we assume $\frac{K}{a} =: J \in \mathbb{N}$ for simplicity and use $\lambda = \frac{M}{a}$ without loss of generality. The set of all $\boldsymbol{w}$ satisfying the condition in (3) is an *oblique manifold* $\mathcal{M}$ that is the *direct product of spheres* [5]. Since $\|\boldsymbol{w}\|_2^2 = 1$ for all $\boldsymbol{w} \in \mathcal{M}$, the proposed window $\boldsymbol{w}_{\mathcal{M},p}$ is a solution to the problem below

$$\underset{\boldsymbol{w} \in \mathcal{M}}{\mathrm{maximize}} \; \boldsymbol{w}^{\mathrm{T}} \boldsymbol{Q}_p \boldsymbol{w}. \tag{4}$$

We solve the problem in (4) by the Riemannian Newton's method [5], where the cost function is redefined as $h(\boldsymbol{w}) := \frac{1}{2} \boldsymbol{w}^{\mathrm{T}} \boldsymbol{Q}_p \boldsymbol{w}$. We iterate

$$\boldsymbol{w}^{(i+1)} = P_{\mathcal{M}}\big(\boldsymbol{w}^{(i)} - \boldsymbol{H}_{\boldsymbol{w}^{(i)}}^{-1} \boldsymbol{g}_{\boldsymbol{w}^{(i)}}\big) \tag{5}$$

from $\boldsymbol{w}^{(0)} \in \mathcal{M}$, where $P_{\mathcal{M}} : \boldsymbol{w} \mapsto \sqrt{\lambda}\, \boldsymbol{S}_{\boldsymbol{w}}^{-1/2} \boldsymbol{w}$ is the metric projection onto $\mathcal{M}$ also used in (1), and $\boldsymbol{g}_{\boldsymbol{w}} \in \mathbb{R}^K$ and $\boldsymbol{H}_{\boldsymbol{w}} \in \mathbb{R}^{K\times K}$ are the *Riemannian gradient and Hessian of* $h(\boldsymbol{w})$ *at* $\boldsymbol{w} \in \mathcal{M}$ given by

$$\begin{cases} \boldsymbol{g}_{\boldsymbol{w}} = \left(\boldsymbol{Q}_p - a\,\mathrm{diag}(h_l(\boldsymbol{w}))_{l=0}^{K-1}\right)\boldsymbol{w}, \\ \boldsymbol{H}_{\boldsymbol{w}} = \boldsymbol{Q}_p - a\,\mathrm{diag}(h_l(\boldsymbol{w}))_{l=0}^{K-1} - a\boldsymbol{W}\boldsymbol{W}^{\mathrm{T}}\boldsymbol{Q}_p. \end{cases}$$

Here $h_l(\boldsymbol{w}) := \sum_{n=-\lfloor l/a \rfloor}^{\lfloor (K-l-1)/a \rfloor} w[l+an]q[l+an]$ is defined with $\boldsymbol{Q}_p\boldsymbol{w} =: (q[0], q[1], \ldots, q[K-1])^{\mathrm{T}}$, and $\boldsymbol{W} := (\mathrm{diag}(w[l])_{l=0}^{a-1}, \mathrm{diag}(w[l+a])_{l=0}^{a-1}, \ldots, \mathrm{diag}(w[l+a(J-1)])_{l=0}^{a-1})^{\mathrm{T}} \in \mathbb{R}^{K\times a}$ is given from $\boldsymbol{w}$ (see [6] for details). An equivalent algorithm is used for the *multivariate eigenvalue problem* [7].

4 Numerical Experiments

We show windows designed for $K = 512$, $a = 128$ and $p \in \{\frac{1}{K}, \frac{2}{K}, \ldots, \frac{20}{K}\}$. For $p = \frac{1}{K}$, the initial value $\boldsymbol{w}^{(0)}$ of the proposed algorithm was set to the tight window $\sqrt{\lambda}\, \boldsymbol{S}_{\boldsymbol{w}_{\mathcal{S},p}}^{-1/2} \boldsymbol{w}_{\mathcal{S},p}$ given from the Slepian window $\boldsymbol{w}_{\mathcal{S},p}$. Then, for $p = \frac{n}{K}$ $(n \geq 2)$, the proposed window $\boldsymbol{w}_{\mathcal{M},p}$ for $p = \frac{n-1}{K}$ was used as the initial value. The iteration in (5) was terminated when $\|\boldsymbol{g}_{\boldsymbol{w}^{(i)}}\|_2 \leq 10^{-15}$ was satisfied.

The value of $\|\boldsymbol{g}_{\boldsymbol{w}^{(i)}}\|_2$ at each iteration is shown in Fig. 1, where each line corresponds to one of the mainlobe-width parameters p (see Table 1 for each color), and the stopping criterion $\delta = 10^{-15}$ is indicated by the horizontal line. Table 1 shows the numbers of iterations required for the termination. The proposed algorithm could compute the solution very fast for $p \leq \frac{13}{K}$ while required more iterations for $p \geq \frac{14}{K}$, but it was still fast except for $p = \frac{19}{K}$. This instability is because the eigenvalues of $\boldsymbol{Q}_p$ are closer to each other as p increases [3].

Fig. 2 shows shapes and spectra of the obtained windows. The Slepian window $\boldsymbol{w}_{\mathcal{S},p}$ (top row) becomes narrower as p increases, which implies the increase of the distance from the set $\mathcal{M}$. Hence, its canonical tight window $\sqrt{\lambda}\, \boldsymbol{S}_{\boldsymbol{w}_{\mathcal{S},p}}^{-1/2} \boldsymbol{w}_{\mathcal{S},p}$ (middle row) is more different from $\boldsymbol{w}_{\mathcal{S},p}$ for larger p, and the energy-concentration property is broken as in the middle right figure. In contrast, the proposed window $\boldsymbol{w}_{\mathcal{M},p}$ (bottom row) can narrow the mainlobe as $\boldsymbol{w}_{\mathcal{S},p}$ while satisfying the tightness in (3).

5 Conclusion

In this paper, we proposed a class of tight windows minimizing the sidelobe energy. Those windows are characterized as solutions to the spectral concentration problems on oblique manifolds. We exactly applied the Riemannian Newton's method for computing the solutions fast. Improvement of the numerical stability for large p and applications of the proposed tight windows to STFT/DGT domain signal processing are left as future work.

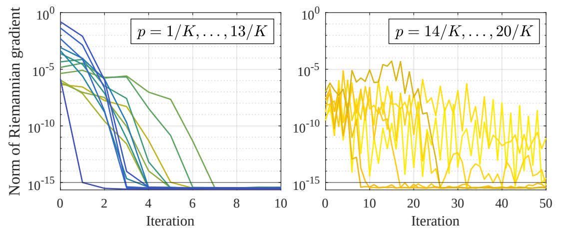

Fig. 1. Values of $\|\boldsymbol{g}_{\boldsymbol{w}^{(i)}}\|_2$. Since the numbers of iterations required for convergence were largely different for some p (see Table 1), the figure was split into two parts: results for $p = \frac{1}{K}, \frac{2}{K}, \ldots, \frac{13}{K}$ (left) and for $p = \frac{14}{K}, \frac{15}{K}, \ldots, \frac{20}{K}$ (right). The color represents p, e.g., $\frac{1}{K}$ is dark blue and $\frac{20}{K}$ is yellow.

Table 1. Numbers of iterations required for convergence.

p	i	p	i	p	i	p	i
$1/K$	2	$6/K$	4	$11/K$	5	$16/K$	10
$2/K$	4	$7/K$	4	$12/K$	4	$17/K$	17
$3/K$	3	$8/K$	5	$13/K$	6	$18/K$	26
$4/K$	3	$9/K$	6	$14/K$	26	$19/K$	266
$5/K$	4	$10/K$	7	$15/K$	8	$20/K$	43

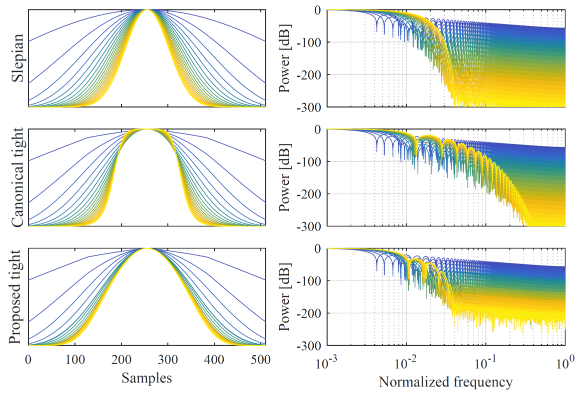

Fig. 2. Comparison of the existing and proposed windows ($K = 512$, $a = 128$ and $p = \frac{1}{K}, \frac{2}{K}, \ldots, \frac{20}{K}$). From top to bottom, the Slepian window, its canonical tight window, and the proposed tight window are shown. The color represents p as in Fig. 1. All lines are peak-normalized, and the frequency axis is normalized so that the Nyquist frequency equals 10^0.

References

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