ONE-DIMENSIONAL EDGE-PRESERVING SPLINE SMOOTHING FOR ESTIMATION OF PIECEWISE SMOOTH FUNCTIONS

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ABSTRACT

Splines are piecewise polynomials and widely used for interpolation and smoothing of observed data, due to their flexibility and optimality in the sense of certain variational problems for one-dimensional (1D) data. However, spline interpolation and smoothing are applicable only to the estimation of *continuous functions*, and not suitable for that of *piecewise smooth functions*. In this paper, we propose a novel spline smoothing technique for the estimation of 1D piecewise smooth functions. We newly define the set of *breaking splines*, which are permitted to have several discontinuous knots. Then, we estimate a piecewise smooth function as a breaking spline minimizing the sum of the data fidelity term, the roughness penalty term, and the number of the discontinuous knots. Numerical experiments show the effectiveness of the breaking splines compared to the conventional splines and the state-of-the-art *total generalized variation (TGV)* denoising.

Index Terms— Spline, piecewise smooth function, function estimation, edge-preserving smoothing, convex relaxation.

1. INTRODUCTION

Let $f_j : (\zeta_{j-1}, \zeta_j) \to \mathbb{R}$ $(j = 1, 2, \ldots, m)$ be ρ-times continuously differentiable functions, i.e., $f_j \in C^\rho(\zeta_{j-1}, \zeta_j)$, where $\rho \in \mathbb{N} \cup \{\infty\}$ and $-\infty \leq \zeta_0 < \zeta_1 < \cdots < \zeta_m \leq \infty$. Define a *piecewise smooth function* $f : (\zeta_0, \zeta_m) \to \mathbb{R}$ by

$$f(x) := f_j(x) \qquad \text{for } x \in (\zeta_{j-1}, \zeta_j). \tag{1}$$

As the function values of f at ζ_j $(j = 1, 2, \ldots, m-1)$, although several cases such as $\lim_{x\to\zeta_j-0} f_j(x)$, $\lim_{x\to\zeta_j+0} f_{j+1}(x)$, and $\lim_{\substack{x\to\zeta_j-0\\ y\to\zeta_j+0}} \frac{1}{2}(f_j(x) + f_{j+1}(y))$ can be considered, we do not take care about them in this paper. Alternatively, we suppose the piecewise function f is strictly discontinuous at ζ_j $(j = 1, 2, \ldots, m-1)$, i.e., $\lim_{x\to\zeta_j-0} f_j(x) \neq \lim_{x\to\zeta_j+0} f_{j+1}(x)$ holds, and we also suppose $\lim_{x\to\zeta_j-0} |f_j^{(l)}(x)| < \infty$ and $\lim_{x\to\zeta_j+0} |f_{j+1}^{(l)}(x)| < \infty$ hold for $l = 0, 1, \ldots, \rho$. We observe finite noisy samples of f by

$$z_i := f(x_i) + \epsilon_i \qquad (i = 1, 2, \ldots, n), \tag{2}$$

where $\epsilon_i \in \mathbb{R}$ is additive white Gaussian noise, and sampling points x_i satisfy $x_1 > \zeta_0$, $x_n < \zeta_m$, $\forall i\; x_{i+1} - x_i = h > 0$, $\forall i\,\forall j\; x_i \neq \zeta_j$ and $\forall j\,\exists i\; x_i \in (\zeta_{j-1}, \zeta_j)$. In this paper, we address an estimation problem of the piecewise smooth function f in (1) from its noisy samples in (2). Such a problem appears in wide areas from science to engineering [1]–[10]. In [11], the authors assume that all pieces f_j can be expressed as $f_j(x) = \sum_{k=1}^K c_k^{\langle j\rangle} \phi_k(x)$ with the use of *known* common basis functions $\{\phi_k\}_{k=1}^K$. On the other hand, in this paper, we assume that such basis functions are *unknown* or *do not exist*.

This work was supported in part by JSPS Grants-in-Aid Grant Number JP17H07243 (e-mail: d-kita@fc.ritsumei.ac.jp).

A spline is a function that is piecewise-defined by polynomials, and can possess certain-times continuous differentiability, including at locations where the polynomial pieces connect. Splines have been widely used for interpolation and smoothing of data in many signal and image processing areas [12] such as super-resolution [13], [14], computer aided design [15], [16], and regression analysis [17], [18]. The most commonly used splines are *cubic splines*, i.e., univariate spline functions which are expressed, on sub-intervals, as polynomials of degree 3 at most. This is because cubic splines are the unique solutions of the following variational problems on one-dimensional (1D) interpolation and smoothing [19]–[22].

Problem 1 (A Variational Problem on 1D Interpolation) *Find* $g^* \in C^2(-\infty, \infty)$ *minimizing*

$$\int_{-\infty}^{\infty} |g''(x)|^2 \, \mathrm{d}x \tag{3}$$

subject to

$$g(x_i) = z_i \text{ for all } i = 1, 2, \ldots, n.$$

Problem 2 (A Variational Problem on 1D Smoothing) *Find* $g^* \in C^2(-\infty, \infty)$ *minimizing*

$$\sum_{i=1}^n |g(x_i) - z_i|^2 + \lambda \int_{-\infty}^{\infty} |g''(x)|^2 \, \mathrm{d}x, \tag{4}$$

where the smoothing parameter $\lambda > 0$ *controls the trade-off between the data fidelity and the smoothness. Note that Problem 2 is a generalization of Problem 1 because the solution of Problem 2 approaches that of Problem 1 as* λ *approaches* $+0$.

Problem 1 is called *spline interpolation*, and it is especially effective if noise-free data are available [13]–[16]. Problem 2 is called *spline smoothing*, and it is often used for the designs of continuous models from noisy data [17], [18]. However, spline interpolation and spline smoothing consider only the estimation of *continuous functions*, and not suitable for that of *piecewise smooth functions*, having discontinuous points ζ_j, due to the Gibbs phenomenon in spline interpolation or over-smoothing of edges in spline smoothing.

For the estimation of piecewise smooth functions, a discrete approach is often adopted [23]–[31]. In such an approach, we estimate only the finite function values $f(x_i)$ $(i = 1, 2, \ldots, n)$ by suppressing the noise ϵ_i while preserving the edges. A famous method is the *total variation (TV)* denoising [23], [24]. TV is defined as the absolute sum of discrete gradients, and TV denoising estimates $f(x_i)$ by minimizing the sum of the data fidelity term and TV. Although TV denoising has been widely used in image processing and computer vision areas, it is well-known that the *staircasing effect*, which is the undesirable appearance of small edges, accompanies the use of TV. This is because TV denoising constructs piecewise constant signals, i.e., implicitly assumes all f_j are constant functions s.t. $f_j(x) = c^{\langle j\rangle}$.

To overcome the limitation of TV, the *total generalized variation (TGV)* was proposed as a high order generalization of TV [25], [26]. The kth order TGV of a vector is defined by decomposing the vector into k vectors corresponding to the computation from the first to kth discrete gradient magnitudes (see Section 4 for more detail). In particular, the second order TGV has been used for the regularization in various areas [27]–[31]. The kth order TGV denoising reconstructs piecewise polynomials of degree k, e.g., the second order and third order TGV denoising return piecewise linear and quadratic signals, respectively [25]. As a result, TGV denoising implicitly assumes all f_j are polynomials of degree k at most. Moreover, since the number of parameters increases as the order becomes higher, the fourth and higher order TGV denoising have been hardly used in applications.

In this paper, we propose *edge-preserving spline smoothing* for the estimation of piecewise smooth functions. In Section 2, we newly define *breaking splines* that are splines permitted to have several discontinuous knots. The proposed piecewise smooth function estimation is explained in Section 3. By assuming the sampling interval h is short enough compared with the length of each interval (ζ_{j-1}, ζ_j), we can consider that discontinuous knots of an appropriate breaking spline sparsely exist. We estimate f in (1) by a breaking spline minimizing the sum of the data fidelity, the roughness penalty, and a convex relaxation of the number of the discontinuous knots. In Section 4, numerical experiments show that the proposed method can carry out *edge detection* and *smoothing for other than edges* at the same time.

2. PRELIMINARIES

2.1. Notation

Let $\mathbb{R}$ and $\mathbb{N}$ be the sets of all real numbers and non-negative integers, respectively. For any $\rho \in \mathbb{N} \cup \{\infty\}$ and any open interval $(a, b) \subset \mathbb{R}$, $C^\rho(a, b)$ stands for the set of all ρ-times continuously differentiable real-valued functions on (a, b). For any $d \in \mathbb{N}$, $\mathbb{P}_d (\subset C^\infty(-\infty, \infty))$ stands for the set of all univariate real polynomials of degree d at most, i.e., $\mathbb{P}_d := \{p : \mathbb{R} \to \mathbb{R} : x \mapsto \sum_{k=0}^d c_k x^k \mid c_k \in \mathbb{R}\}$. A boldface small letter expresses a vector, and a boldface capital letter expresses a matrix. For any vector $\boldsymbol{x} := (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, the ℓ_2 and ℓ_1 norms of $\boldsymbol{x}$ are denoted by $\|\boldsymbol{x}\|_2 := \sqrt{\sum_{i=1}^n x_i^2}$ and $\|\boldsymbol{x}\|_1 := \sum_{i=1}^n |x_i|$, respectively.

2.2. Breaking Spline: Spline Having Discontinuous Knots

Let $\sqcup_n := \{I_i := (\xi_{i-1}, \xi_i)\}_{i=1}^n$ be a set of n sub-intervals I_i on an open interval $I := (\xi_0, \xi_n)$, where *knots* satisfy $\xi_i - \xi_{i-1} = h_i > 0$ $(i = 1, 2, \ldots, n)$. For $\sqcup_n$ and any $\rho, d \in \mathbb{N}$ s.t. $0 \le \rho < d$, we define

$$\mathcal{BS}_d^\rho(\sqcup_n) := \left\{ s : (\xi_0, \xi_n) \to \mathbb{R} \;\middle|\; \begin{array}{l} s = p_i \in \mathbb{P}_d \text{ on } I_i, \\ s = \frac{1}{2}(p_i + p_{i+1}) \text{ at } \xi_i, \\ \text{and } s \in C^0(\xi_{i-1}, \xi_{i+1}) \\ \quad \Rightarrow s \in C^\rho(\xi_{i-1}, \xi_{i+1}) \end{array} \right\} \quad (5)$$

as the set of all *breaking splines*, which are *splines permitted to have discontinuous knots* ξ_i, of degree d and smoothness ρ on $\sqcup_n$. In this paper, we express a breaking spline $s \in \mathcal{BS}_d^\rho(\sqcup_n)$ in the following interval normalization form:

$$s(x) := p_i(x) := \sum_{k=0}^d c_k^{\langle i \rangle} \left(\frac{x - \xi_{i-1}}{h_i} \right)^k \quad \text{for } x \in (\xi_{i-1}, \xi_i), \quad (6)$$

where $c_k^{\langle i \rangle} \in \mathbb{R}\ (k = 0, 1, \ldots, d)$ are coefficients of each polynomial $p_i \in \mathbb{P}_d$. Note that the set $\mathcal{BS}_d^\rho(\sqcup_n)$ is not closed and not convex because of the condition $s \in C^0(\xi_{i-1}, \xi_{i+1}) \Rightarrow s \in C^\rho(\xi_{i-1}, \xi_{i+1})$.

2.2.1. Quadratic Form of the Roughness Penalty Term

By restricting the domain of interest to $I = (\xi_0, \xi_n)$ and the function space to $\mathcal{BS}_d^\rho(\sqcup_n)$, the roughness penalty term used in (3) and (4) is expressed as

$$\int_I |s''(x)|^2 \, \mathrm{d}x = \sum_{i=1}^n \int_{I_i} |s''(x)|^2 \, \mathrm{d}x. \quad (7)$$

By using the expression in (6), the roughness penalty on I_i can be expressed as the following quadratic form:

$$\begin{aligned} &\int_{I_i} |s''(x)|^2 \, \mathrm{d}x \\ &= \sum_{k=2}^d \sum_{l=2}^d \frac{k(k-1)l(l-1) c_k^{\langle i \rangle} c_l^{\langle i \rangle}}{h_i^4} \int_{I_i} \left(\frac{x - \xi_{i-1}}{h_i} \right)^{k+l-4} \mathrm{d}x \\ &= \sum_{k=2}^d \sum_{l=2}^d \frac{k(k-1)l(l-1)}{h_i^3 (k+l-3)} c_k^{\langle i \rangle} c_l^{\langle i \rangle} \\ &= \sum_{k=0}^{d-2} \sum_{l=0}^{d-2} \frac{(d-k)(d-k-1)(d-l)(d-l-1)}{h_i^3 (2d-k-l-3)} c_{d-k}^{\langle i \rangle} c_{d-l}^{\langle i \rangle} \\ &= \boldsymbol{c}_i^{\mathrm{T}} \boldsymbol{Q}_i \boldsymbol{c}_i, \end{aligned} \quad (8)$$

where $\boldsymbol{c}_i := (c_d^{\langle i \rangle}, c_{d-1}^{\langle i \rangle}, \ldots, c_0^{\langle i \rangle})^{\mathrm{T}} \in \mathbb{R}^{d+1}$ and a symmetric positive semidefinite matrix $\boldsymbol{Q}_i \in \mathbb{R}^{(d+1)\times(d+1)}$ is defined by

$$[\boldsymbol{Q}_i]_{k+1,l+1} := \frac{(d-k)(d-k-1)(d-l)(d-l-1)}{h_i^3 (2d-k-l-3)} \quad (k = 0, 1, \ldots, d-2 \text{ and } l = 0, 1, \ldots, d-2)$$

and $[\boldsymbol{Q}_i]_{k+1,l+1} := 0$ ($k = d-1, d$ or $l = d-1, d$). From (7) and (8), the roughness penalty on I can be expressed as

$$\int_I |s''(x)|^2 \, \mathrm{d}x = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{Q} \boldsymbol{c}, \quad (9)$$

where $\boldsymbol{c} := (\boldsymbol{c}_1^{\mathrm{T}}, \boldsymbol{c}_2^{\mathrm{T}}, \ldots, \boldsymbol{c}_n^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{n(d+1)}$ is the coefficient vector of $s \in \mathcal{BS}_d^\rho(\sqcup_n)$ and $\boldsymbol{Q} \in \mathbb{R}^{n(d+1)\times n(d+1)}$ is a symmetric positive semidefinite matrix defined by $\boldsymbol{Q} := \mathrm{diag}(\boldsymbol{Q}_1, \boldsymbol{Q}_2, \ldots, \boldsymbol{Q}_n)$.

2.2.2. Linear Equation for the ρ-Times Differentiability

For a breaking spline $s \in \mathcal{BS}_d^\rho(\sqcup_n)$ in (5), to ensure the ρ-times continuous differentiability over (ξ_{i-1}, ξ_{i+1}), i.e., $s \in C^\rho(\xi_{i-1}, \xi_{i+1})$, the coefficients of the adjacent polynomials p_i and p_{i+1} in (6) must satisfy the following equations:

$$\begin{aligned} & s \in C^\rho(\xi_{i-1}, \xi_{i+1}) \\ & \Leftrightarrow p_i^{(l)}(\xi_i) = p_{i+1}^{(l)}(\xi_i) \quad (l = 0, 1, \ldots, \rho) \\ & \Leftrightarrow \frac{1}{h_i^l} \sum_{k=l}^d \frac{k!}{(k-l)!} c_k^{\langle i \rangle} = \frac{l!}{h_{i+1}^l} c_l^{\langle i+1 \rangle} \quad (l = 0, 1, \ldots, \rho) \\ & \Leftrightarrow \frac{1}{h_i^l} \sum_{k=l}^d \frac{k!}{(k-l)!} c_k^{\langle i \rangle} - \frac{l!}{h_{i+1}^l} c_l^{\langle i+1 \rangle} = 0 \quad (l = 0, 1, \ldots, \rho). \end{aligned} \quad (10)$$

From (10), there is a matrix $\boldsymbol{H}_i \in \mathbb{R}^{(\rho+1)\times 2(d+1)}$ satisfying

$$s \in C^\rho(\xi_{i-1}, \xi_{i+1}) \;\Leftrightarrow\; \boldsymbol{H}_i \begin{bmatrix} \boldsymbol{c}_i \\ \boldsymbol{c}_{i+1} \end{bmatrix} = \boldsymbol{0}. \quad (11)$$

In this paper, to remove the ambiguity of $\boldsymbol{H}_i$ on (i) constant multiplication and (ii) the order of the row vectors, we assume that each matrix $\boldsymbol{H}_i$ $(i = 1, 2, \ldots, n-1)$ satisfies

$$\boldsymbol{H}_i \begin{bmatrix} \boldsymbol{c}_i \\ \boldsymbol{c}_{i+1} \end{bmatrix} = \begin{bmatrix} p_i(\xi_i) - p_{i+1}(\xi_i) \\ p_i'(\xi_i) - p_{i+1}'(\xi_i) \\ \vdots \\ p_i^{(\rho)}(\xi_i) - p_{i+1}^{(\rho)}(\xi_i) \end{bmatrix}. \tag{12}$$

3. PIECEWISE SMOOTH FUNCTION ESTIMATION BY EDGE-PRESERVING SPLINE SMOOTHING

In this section, we estimate the piecewise smooth function f in (1) from its noisy samples z_i in (2) with the use of a breaking spline $s \in \mathcal{BS}_d^\rho(\sqcup_n)$, where the knots are defined by $\xi_0 := x_1 - h/2$ and $\xi_i := x_i + h/2$ $(i = 1, 2, \ldots, n)$. Then from (6), the function values of s at the sampling points $x_i \in (\xi_{i-1}, \xi_i)$ $(i = 1, 2, \ldots, n)$ are given by

$$s(x_i) = \sum_{k=0}^{d} c_k^{\langle i \rangle} \frac{1}{2^k} = \begin{bmatrix} \frac{1}{2^d} & \frac{1}{2^{d-1}} & \cdots & 1 \end{bmatrix} \boldsymbol{c}_i =: \boldsymbol{a}^{\mathrm{T}} \boldsymbol{c}_i. \tag{13}$$

Therefore, the data fidelity term used in (4) can be expressed as

$$\sum_{i=1}^{n} |s(x_i) - z_i|^2 = \|\boldsymbol{A}\boldsymbol{c} - \boldsymbol{z}\|_2^2, \tag{14}$$

where $\boldsymbol{z} := (z_1, z_2, \ldots, z_n)^{\mathrm{T}} \in \mathbb{R}^n$ and $\boldsymbol{A} \in \mathbb{R}^{n \times n(d+1)}$ is a matrix whose row vectors are based on the vector $\boldsymbol{a}^{\mathrm{T}}$ in (13). Note that for conventional splines, i.e., the solutions of Problems 1 and 2, each knot ξ_i is located at the sampling point x_i while for the proposed splines, each knot ξ_i is located at the middle point of x_i and x_{i+1}, and hence the ith sample z_i is related only to the ith polynomial p_i.

Suppose that the sampling interval h is short enough compared with the length of each (ζ_{j-1}, ζ_j) $(j = 1, 2, \ldots, m)$, i.e., there are enough samples to accurately reconstruct each f_j if the discontinuous points ζ_j can be detected. Then, the discontinuous knots ξ_i of $s \in \mathcal{BS}_d^\rho(\sqcup_n)$, approximating f, should sparsely exist on (ξ_0, ξ_n). Moreover, even if $\lim_{\substack{x \to \zeta_j - 0 \\ y \to \zeta_j + 0}} |f_j^{(l)}(x) - f_{j+1}^{(l)}(y)| \not\approx 0$ $(l = 0, 1, 2)$ and $x_i < \zeta_j < x_{i+1}$ hold for some i and j, $|s(x_i) - z_i|^2$, $|s(x_{i+1}) - z_{i+1}|^2$ and $\int_{\xi_{i-1}}^{\xi_{i+1}} |s''(x)|^2 \, \mathrm{d}x$ can be small for $s \notin C^0(\xi_{i-1}, \xi_{i+1})$.

On the basis of the above discussion, we propose to solve the following *non-convex* optimization problem based on Problem 2:

$$\underset{s \in \mathcal{BS}_d^\rho(\sqcup_n)}{\text{minimize}} \sum_{i=1}^{n} |s(x_i) - z_i|^2 + \lambda \int_I |s''(x)|^2 \, \mathrm{d}x + \kappa \mathrm{ND}(s), \tag{15}$$

where $\rho \geq 2$, $\lambda > 0$, $\kappa > 0$, and $\mathrm{ND}(s) \in \mathbb{N}$ denotes the number of discontinuous knots ξ_i s.t. $p_i(\xi_i) \neq p_{i+1}(\xi_i)$. From (9), (11), and (14), the problem in (15) can be expressed as an optimization problem on the coefficient vector $\boldsymbol{c} \in \mathbb{R}^{n(d+1)}$:

$$\underset{\boldsymbol{c} \in \mathbb{R}^{n(d+1)}}{\text{minimize}} \|\boldsymbol{A}\boldsymbol{c} - \boldsymbol{z}\|_2^2 + \lambda \boldsymbol{c}^{\mathrm{T}} \boldsymbol{Q} \boldsymbol{c} + \kappa \sum_{i=1}^{n-1} \Gamma \left(\boldsymbol{H}_i \begin{bmatrix} \boldsymbol{c}_i \\ \boldsymbol{c}_{i+1} \end{bmatrix} \right), \tag{16}$$

where $\Gamma : \mathbb{R}^{\rho+1} \to \{0, 1\}$ is a binary function defined by $\Gamma(\boldsymbol{x}) := 0$ if $\boldsymbol{x} = \boldsymbol{0}$, and $\Gamma(\boldsymbol{x}) := 1$ otherwise. In order to approximately solve the problem in (16), we use a convex relaxation technique.

The third term of the cost function in (16) can be considered as a group ℓ_0 (pseudo) norm without overlapping:

$$\|\boldsymbol{H}\boldsymbol{c}\|_0^{\mathcal{G}} := \sum_{i=1}^{n-1} \Gamma \left(\boldsymbol{H}_i \begin{bmatrix} \boldsymbol{c}_i \\ \boldsymbol{c}_{i+1} \end{bmatrix} \right) \tag{17}$$

with the use of some matrix $\boldsymbol{H} \in \mathbb{R}^{(n-1)(\rho+1) \times n(d+1)}$ based on the matrices $\boldsymbol{H}_i$ $(i = 1, 2, \ldots, n-1)$. Hence, we replace the group ℓ_0 norm in (17) with a *weighted* group ℓ_1 norm:

$$\begin{aligned} \|\boldsymbol{H}\boldsymbol{c}\|_{1,\boldsymbol{w}}^{\mathcal{G}} &:= \sum_{i=1}^{n-1} \left\| \boldsymbol{H}_i \begin{bmatrix} \boldsymbol{c}_i \\ \boldsymbol{c}_{i+1} \end{bmatrix} \right\|_{2,\boldsymbol{w}_i} \\ &:= \sum_{i=1}^{n-1} \sqrt{\sum_{l=0}^{\rho} w_l^{\langle i \rangle} \left| p_i^{(l)}(\xi_i) - p_{i+1}^{(l)}(\xi_i) \right|^2}, \end{aligned} \tag{18}$$

where $w_l^{\langle i \rangle} > 0$, $\boldsymbol{w}_i := (w_0^{\langle i \rangle}, w_1^{\langle i \rangle}, \ldots, w_\rho^{\langle i \rangle})^{\mathrm{T}} \in \mathbb{R}^{\rho+1}$, and $\boldsymbol{w} := (\boldsymbol{w}_1^{\mathrm{T}}, \boldsymbol{w}_2^{\mathrm{T}}, \ldots, \boldsymbol{w}_{n-1}^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{(n-1)(\rho+1)}$. In (18), we use the definitions of the matrices $\boldsymbol{H}_i$ in (12). As a result, we solve the following *convex* optimization problem:

$$\underset{\boldsymbol{c} \in \mathbb{R}^{n(d+1)}}{\text{minimize}} \|\boldsymbol{A}\boldsymbol{c} - \boldsymbol{z}\|_2^2 + \lambda \boldsymbol{c}^{\mathrm{T}} \boldsymbol{Q} \boldsymbol{c} + \kappa \|\boldsymbol{H}\boldsymbol{c}\|_{1,\boldsymbol{w}}^{\mathcal{G}} \tag{19}$$

for the estimation of the piecewise smooth function f. We name the problems in (15) and (19) "*edge-preserving spline smoothing*." The optimal solution of the problem in (19) can be computed by the alternating direction method of multipliers (ADMM) [32].

ADMM solves the following optimization problem:

$$\text{find } \boldsymbol{x}^* \in \underset{\boldsymbol{x} \in \mathbb{R}^n}{\operatorname{argmin}} \; F(\boldsymbol{x}) + G(\boldsymbol{L}\boldsymbol{x}),$$

where $\boldsymbol{L} \in \mathbb{R}^{m \times n}$ and two functions $F : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ and $G : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ are *proper, lower semicontinuous* and *convex*.[1] From any initialization of $(\boldsymbol{\mu}_0, \boldsymbol{\nu}_0) \in \mathbb{R}^m \times \mathbb{R}^m$, ADMM iteratively computes

$$\left| \begin{aligned} \boldsymbol{x}_{t+1} &= \underset{\boldsymbol{x} \in \mathbb{R}^n}{\operatorname{argmin}} \; F(\boldsymbol{x}) + \frac{1}{2\gamma} \|\boldsymbol{\mu}_t - \boldsymbol{L}\boldsymbol{x} - \boldsymbol{\nu}_t\|_2^2 \\ \boldsymbol{\mu}_{t+1} &= \mathrm{prox}_{\gamma G}(\boldsymbol{L}\boldsymbol{x}_{t+1} + \boldsymbol{\nu}_t) \\ \boldsymbol{\nu}_{t+1} &= \boldsymbol{\nu}_t + \boldsymbol{L}\boldsymbol{x}_{t+1} - \boldsymbol{\mu}_{t+1} \end{aligned} \right. \tag{20}$$

for $t \geq 0$ with any $\gamma > 0$, where $\mathrm{prox}_{\gamma G} : \mathbb{R}^m \to \mathbb{R}^m$ denotes the *proximity operator*[2] of γG. Then $(\boldsymbol{x}_t)_{t=1}^\infty$ converges to the optimal solution $\boldsymbol{x}^*$. To solve the problem in (19), we define $\boldsymbol{L} := \boldsymbol{W}\boldsymbol{H}$, $F(\boldsymbol{c}) := \|\boldsymbol{A}\boldsymbol{c} - \boldsymbol{z}\|_2^2 + \lambda \boldsymbol{c}^{\mathrm{T}} \boldsymbol{Q} \boldsymbol{c}$, and $G(\boldsymbol{L}\boldsymbol{c}) := \kappa \|\boldsymbol{L}\boldsymbol{c}\|_1^{\mathcal{G}}$, where $\boldsymbol{W} \in \mathbb{R}^{(n-1)(\rho+1) \times (n-1)(\rho+1)}$ is a diagonal matrix whose components are the square roots of $w_l^{\langle i \rangle}$. Moreover, by redefining 2γ as γ, the iterative computation in (20) is expresses as

$$\left| \begin{aligned} \boldsymbol{c}_{t+1} &= \left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda \boldsymbol{Q} + \frac{1}{\gamma} \boldsymbol{H}^{\mathrm{T}} \boldsymbol{W}^2 \boldsymbol{H} \right)^{-1} \\ &\quad \cdot \left(\boldsymbol{A}^{\mathrm{T}} \boldsymbol{z} + \frac{1}{\gamma} \boldsymbol{H}^{\mathrm{T}} \boldsymbol{W} (\boldsymbol{\mu}_t - \boldsymbol{\nu}_t) \right) \\ \boldsymbol{\mu}_{t+1} &= \mathrm{prox}_{\frac{\gamma\kappa}{2} \|\cdot\|_1^{\mathcal{G}}}(\boldsymbol{W}\boldsymbol{H}\boldsymbol{c}_{t+1} + \boldsymbol{\nu}_t) \\ \boldsymbol{\nu}_{t+1} &= \boldsymbol{\nu}_t + \boldsymbol{W}\boldsymbol{H}\boldsymbol{c}_{t+1} - \boldsymbol{\mu}_{t+1} \end{aligned} \right.$$

[1] A function $F : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is called proper, lower semicontinous, and convex if $\mathrm{dom}(F) := \{\boldsymbol{x} \in \mathbb{R}^n \mid F(\boldsymbol{x}) < \infty\} \neq \varnothing$, $\mathrm{lev}_{\leq \alpha}(F) := \{\boldsymbol{x} \in \mathbb{R}^n \mid F(\boldsymbol{x}) \leq \alpha\}$ is closed for all $\alpha \in \mathbb{R}$, and $F(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \leq \lambda F(\boldsymbol{x}) + (1-\lambda) F(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\lambda \in (0, 1)$, respectively.

[2] For a proper, lower semicontinous, convex function $F : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, the proximity operator $\mathrm{prox}_F : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\mathrm{prox}_F(\boldsymbol{x}) := \underset{\boldsymbol{y} \in \mathbb{R}^n}{\operatorname{argmin}} \; F(\boldsymbol{y}) + \tfrac{1}{2}\|\boldsymbol{y} - \boldsymbol{x}\|_2^2.$$

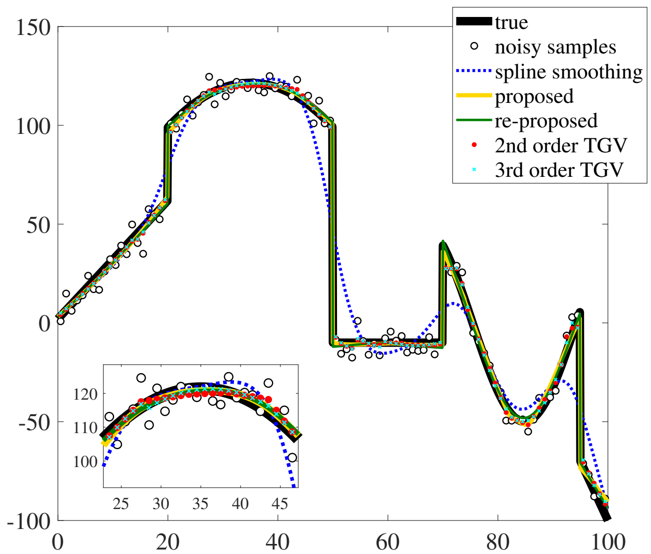

Fig. 1. Experimental results I.

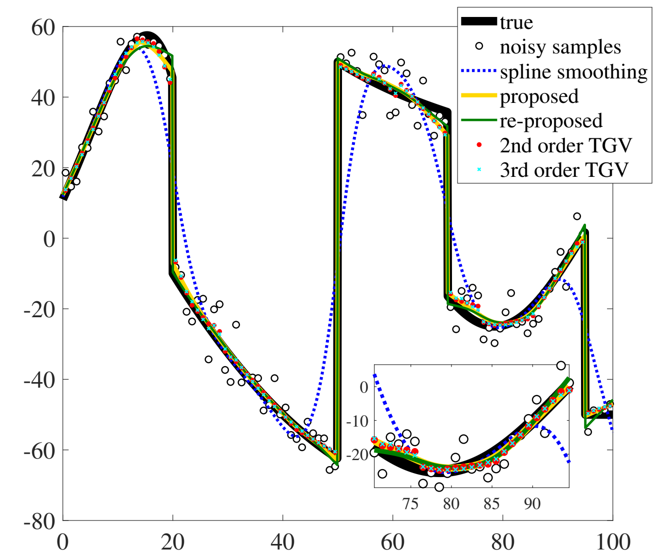

Fig. 2. Experimental results II.

with any $\gamma > 0$ and any $(\boldsymbol{\mu}_0, \boldsymbol{\nu}_0) \in \mathbb{R}^{(n-1)(\rho+1)} \times \mathbb{R}^{(n-1)(\rho+1)}$. Then $(\boldsymbol{c}_t)_{t=1}^{\infty}$ converges to the solution of the problem in (19).

Finally, to obtain a breaking spline $s \in \mathcal{BS}_d^{\rho}(\sqcup_n)$ as an estimate of f, we re-solve the problem in (15). From the optimal coefficient vector $\boldsymbol{c}^*$ of the problem in (19), we detect *continuous knots* ξ_i of $s \in \mathcal{BS}_d^{\rho}(\sqcup_n)$ by checking, for each ξ_i, whether $|p_i^*(\xi_i) - p_{i+1}^*(\xi_i)|$ is lower than a threshold value $\tau > 0$. In (5), if ξ_i is the continuous knot, then the condition $s \in C^{\rho}(\xi_{i-1}, \xi_{i+1})$ must hold. Therefore, to obtain the solution of (15), we solve

$$\begin{aligned} &\underset{\boldsymbol{c}\in\mathbb{R}^{n(d+1)}}{\text{minimize}} \ \|\boldsymbol{A}\boldsymbol{c} - \boldsymbol{z}\|_2^2 + \lambda \boldsymbol{c}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{c} \\ &\text{subject to } \ \boldsymbol{H}_i \begin{bmatrix} \boldsymbol{c}_i \\ \boldsymbol{c}_{i+1} \end{bmatrix} = \boldsymbol{0} \ \text{ for all continuous knots } \xi_i. \end{aligned} \tag{21}$$

The problem in (21) is solved by a quadratic programming solver implemented as "interior-point-convex method" in MATLAB [33], [34].

4. NUMERICAL EXPERIMENTS

Define $\zeta_0 := 0$, $\zeta_1 := 20$, $\zeta_2 := 50$, $\zeta_3 := 70$, $\zeta_4 := 95$, $\zeta_5 := 100$, and $x_i := i - 0.5$ $(i = 1, 2, \ldots, n := 100)$. As a result, knots of a breaking spline $s \in \mathcal{BS}_3^2(\sqcup_n)$ are $\xi_i := i$ $(i = 0, 1, \ldots, 100)$. For two piecewise smooth functions f depicted by black lines in Figs. 1 and 2, we try to reconstruct them from noisy samples z_i in (2), where the standard division of additive white Gaussian noise ϵ_i is $\sigma = 5$. We compare the estimates by the edge-preserving spline smoothing (the problem in (19) followed by that in (21)), the conventional spline smoothing (Problem 2) and the second/third order TGV denoising:

$$\underset{\boldsymbol{r}\in\mathbb{R}^n}{\text{minimize}} \ \|\boldsymbol{r} - \boldsymbol{z}\|_2^2 + \beta\, \mathrm{TGV}_{\boldsymbol{\alpha}}^k(\boldsymbol{r}), \tag{22}$$

where $\boldsymbol{r} := (r_1, r_2, \ldots, r_n)^{\mathrm{T}} \in \mathbb{R}^n$ is an estimate of $f(x_i)$, $\beta > 0$, $k = 2, 3$, and $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \ldots, \alpha_k)^{\mathrm{T}} \in (0, 1)^k$ s.t. $\sum_{i=1}^k \alpha_i = 1$. The second order TGV is defined by

$$\mathrm{TGV}_{\boldsymbol{\alpha}}^2(\boldsymbol{r}) := \min_{\boldsymbol{u}} \ \alpha_1 \|\boldsymbol{D}_1 \boldsymbol{r} - \boldsymbol{u}\|_1 + \alpha_2 \|\boldsymbol{D}_2 \boldsymbol{u}\|_1,$$

and the third order TGV is defined by

$$\mathrm{TGV}_{\boldsymbol{\alpha}}^3(\boldsymbol{r}) := \min_{\boldsymbol{u},\boldsymbol{v}} \ \alpha_1 \|\boldsymbol{D}_1 \boldsymbol{r} - \boldsymbol{u}\|_1 + \alpha_2 \|\boldsymbol{D}_2 \boldsymbol{u} - \boldsymbol{v}\|_1 + \alpha_3 \|\boldsymbol{D}_3 \boldsymbol{v}\|_1$$

with difference matrices $\boldsymbol{D}_1 \in \mathbb{R}^{(n-1)\times n}$, $\boldsymbol{D}_2 \in \mathbb{R}^{(n-2)\times(n-1)}$ and $\boldsymbol{D}_3 \in \mathbb{R}^{(n-3)\times(n-2)}$. We solve the problem in (22) by ADMM. The smoothing parameter is set to $\lambda = 65$. The weights in (19) are set to $w_0^{\langle i \rangle} = \frac{1}{(z_{i+1} - z_i)^2 + 1}$, $w_1^{\langle i \rangle} = 0.01$, $w_2^{\langle i \rangle} = 0.000001$, and $\kappa = 600$. In (22), for $k = 2$, we use $(\alpha_1, \alpha_2) = (0.35, 0.65)$ and $\beta = 35$, and for $k = 3$, we use $(\alpha_1, \alpha_2, \alpha_3) = (0.1, 0.2, 0.7)$ and $\beta = 130$.

In Figs. 1 and 2, black circles denote the observed noisy samples z_i. Blue, yellow, and green lines depict the estimation results of the piecewise function f by the conventional spline smoothing in (4), the proposed edge-preserving spline smoothing in (19), and the proposed re-estimation in (21), respectively. Red circles and light blue crosses denote the estimation results of the function values $f(x_i)$ by the second order and third order TGV denoising in (22), respectively. From Figs. 1 and 2, we can see that the estimation results by the conventional spline smoothing lose the edges of the true piecewise smooth functions because the conventional splines are not permitted to have discontinuous knots. On the other hand, because the breaking splines can express discontinuous points, the estimation results by the proposed edge-preserving spline smoothing in (19) and (21) are really good. In particular, we find that the solution of the problem in (19) is very similar to that of the problem in (21), i.e., the problem in (19) achieves both detection of the discontinuous points ζ_j and smoothing for other than the edges. The estimates by the second order TGV denoising are piecewise linear, and hence the second order TGV cannot restore the smooth pieces while the third order TGV returns smoother estimates, but there still exist several undesirable small edges.

5. CONCLUSION

In this paper, we proposed novel spline smoothing for the estimation of piecewise smooth functions. For this purpose, we defined breaking splines permitted to have several discontinuous knots differently from conventional splines. We estimated piecewise smooth functions by breaking splines which minimize the sum of the data fidelity term, the roughness penalty term, and the number of discontinuous knots. The minimizer is computed by ADMM, and numerical experiments demonstrated the effectiveness of the edge-preserving spline smoothing. In future work, we plan to extend our approach to 2D images.

6. REFERENCES

[1] X. Liu, N. D. Sidiropoulos, and A. Swami, "Blind high-resolution localization and tracking of multiple frequency hopped signals," *IEEE Transactions on Signal Processing*, vol. 50, no. 4, pp. 889–901, Apr. 2002.

[2] A. Valyrakis, E. E. Tsakonas, N. D. Sidiropoulos, and A. Swami, "Stochastic modeling and particle filtering algorithms for tracking a frequency-hopped signal," *IEEE Transactions on Signal Processing*, vol. 57, no. 8, pp. 3108–3118, Aug. 2009.

[3] D. Angelosante, G. B. Giannakis, and N. D. Sidiropoulos, "Estimating multiple frequency-hopping signal parameters via sparse linear regression," *IEEE Transactions on Signal Processing*, vol. 58, no. 10, pp. 5044–5056, Oct. 2010.

[4] D. Angelosante, G. B. Giannakis, and N. D. Sidiropoulos, "Sparse parametric models for robust nonstationary signal analysis: Leveraging the power of sparse regression," *IEEE Signal Processing Magazine*, vol. 30, no. 6, pp. 64–73, 2013.

[5] R. Giryes, M. Elad, and A. M. Bruckstein, "Sparsity based methods for overparameterized variational problems," *SIAM Journal on Imaging Sciences*, vol. 8, no. 3, pp. 2133–2159, 2015.

[6] S. J. Kim, K. Koh, S. Boyd, and D. Gorinevsky, "ℓ_1 trend filtering," *SIAM Review*, vol. 51, no. 2, pp. 339–360, 2009.

[7] G. Steidl, S. Didas, and J. Neumann, "Splines in higher order TV regularization," *International Journal of Computer Vision*, vol. 70, no. 3, pp. 241–255, 2006.

[8] R. J. Tibshirani, "Adaptive piecewise polynomial estimation via trend filtering," *The Annals of Statistics*, vol. 42, no. 1, pp. 285–323, Feb. 2014.

[9] G. Ongie and M. Jacob, "Recovery of discontinuous signals using group sparse higher degree total variation," *IEEE Signal Processing Letters*, vol. 22, no. 9, pp. 1414–1418, Sep. 2015.

[10] I. W. Selesnick, "Generalized total variation: Tying the knots," *IEEE Signal Processing Letters*, vol. 22, no. 11, pp. 2009–2013, Nov. 2015.

[11] H. Kuroda, M. Yamagishi, and I. Yamada, "Recovery of piecewise finite-dimensional continuous signals by exploiting sparsity," in *Proceedings of International Conference on Sampling Theory and Applications (SampTA)*, 2017, pp. 609–613.

[12] M. Unser, "Splines: A perfect fit for signal and image processing," *IEEE Signal Processing Magazine*, vol. 16, no. 6, pp. 22–38, Nov. 1999.

[13] H. S. Hou and H. Andrews, "Cubic splines for image interpolation and digital filtering," *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 26, no. 6, pp. 508–517, Dec. 1978.

[14] R. Keys, "Cubic convolution interpolation for digital image processing," *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 29, no. 6, pp. 1153–1160, Dec. 1981.

[15] E. Cohen, T. Lyche, and R. Riesenfeld, "Discrete B-splines and subdivision techniques in computer-aided geometric design and computer graphics," *Computer Graphics and Image Processing*, vol. 14, no. 2, pp. 87–111, Oct. 1980.

[16] G. Farin, *Curves and Surfaces for Computer-Aided Geometric Design: A Practical Code*, 4th ed. Academic Press, 1996.

[17] S. Wold, "Spline functions in data analysis," *Technometrics*, vol. 16, no. 1, pp. 87–111, Feb. 1974.

[18] L. C. Marsh and D. R. Cormier, *Spline Regression Models*, ser. Quantitative Applications in the Social Sciences. Thousand Oaks, CA, USA: SAGE, 2001, vol. 137.

[19] C. de Boor, "Best approximation properties of spline functions of odd degree," *Journal of Mathematics and Mechanics*, vol. 12, no. 5, pp. 747–749, 1963.

[20] B. W. Silverman, "Some aspects of the spline smoothing approach to non-parametric regression curve fitting," *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, vol. 47, no. 1, pp. 1–52, 1985.

[21] G. Wahba, *Spline Models for Observational Data*, ser. CBMS-NSF Regional Conference Series in Applied Mathematics. Philadelphia, PA, USA: SIAM, 1990, vol. 59.

[22] J. O. Ramsay and B. W. Silverman, *Functional Data Analysis*, 2nd ed. New York, NY, USA: Springer, 2005.

[23] L. I. Rudin, S. Osher, and E. Fatemi, "Nonlinear total variation based noise removal algorithms," *Physica D: Nonlinear Phenomena*, vol. 60, no. 1–4, pp. 259–268, Nov. 1992.

[24] A. Chambolle, V. Caselles, D. Cremers, M. Novaga, and T. Pock, "An introduction to total variation for image analysis," in *Theoretical Foundations and Numerical Methods for Sparse Recovery*, M. Fornasier, Ed. de Gruyter, 2010, pp. 263–340.

[25] K. Bredies, K. Kunisch, and T. Pock, "Total generalized variation," *SIAM Journal on Imaging Sciences*, vol. 3, no. 3, pp. 492–526, Sep. 2010.

[26] S. Setzer, G. Steidl, and T. Teuber, "Infimal convolution regularizations with discrete ℓ_1-type functionals," *Communications in Mathematical Sciences*, vol. 9, no. 3, pp. 797–827, 2011.

[27] K. Bredies, "Recovering piecewise smooth multichannel images by minimization of convex functionals with total generalized variation penalty," in *Efficient Algorithms for Global Optimization Methods in Computer Vision*, A. Bruhn, T. Pock, and X. C. Tai, Ed. Springer, 2011, pp 44–77.

[28] T. Valkonen, K. Bredies, and F. Knoll, "Total generalized variation in diffusion tensor imaging," *SIAM Journal on Imaging Sciences*, vol. 6, no. 1, pp. 487–525, Feb. 2013.

[29] T. Miyata, "L infinity total generalized variation for color image recovery," in *Proceedings of IEEE International Conference on Image Processing (ICIP)*, 2013, pp. 449–453.

[30] S. Ono and I. Yamada, "Decorrelated vectorial total variation," in *Proceedings of IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*, 2014, pp. 4090–4097.

[31] S. Ono and I. Yamada, "Second-order total generalized variation constraint," in *Proceedings of IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2014, pp. 4938–4942.

[32] D. Gabay and B. Mercier, "A dual algorithm for the solution of nonlinear variational problems via finite element approximation," *Computers & Mathematics with Applications*, vol. 2, no. 1, pp. 17–40, 1976.

[33] R. J. Vanderbei and T. J. Carpenter, "Symmetric indefinite systems for interior point methods," *Mathematical Programming*, vol. 58, no. 1–3, pp. 1–32, Jan. 1993.

[34] A. Altman and J. Gondzio, "Regularized symmetric indefinite systems in interior point methods for linear and quadratic optimization," *Optimization Methods and Software*, vol. 11, no. 1–4, pp. 275–302, 1999.