A CONVEX LIFTING APPROACH TO IMAGE PHASE UNWRAPPING

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ABSTRACT

The nonlinear inverse problem of 2-D phase unwrapping consists in estimating an image, while its pixel values are observed modulo 2π. A variational formulation is considered, which consists in minimizing an energy, convex or not, under the nonconvex data fidelity constraints. We propose a new convex relaxation of this combinatorial problem. It shows similar or better performances than the state of the art.

***Index Terms*—** phase unwrapping, convex optimization, variational method, convex relaxation, lifting

1. INTRODUCTION AND PROBLEM FORMULATION

Phase unwrapping is a classical imaging problem [1, 2, 3, 4, 5] with a wide range of applications, such as interferometric synthetic aperture radar (InSAR) [6, 7], magnetic resonance imaging [8, 9], interferometry [10], or profilometry [11]. In these applications, the true phase values are observed modulo 2π and lie in the range $[-\pi, \pi)$. Phase unwrapping then consists in recovering the lost integer multiples of 2π, by assuming typically that the sought image is smooth, except at discontinuities involving a small subset of all pixels.

The phase unwrapping problem can be formulated as follows: we want to estimate an unknown image $x^\sharp \in \mathbb{R}^{N_1 \times N_2}$ of height N_1 and width N_2, from its wrapped version

$$y = (x^\sharp)_w, \tag{1}$$

where the wrapping operator, applied pixelwise, maps $t \in \mathbb{R}$ to

$$(t)_w = \big((t + \pi) \bmod 2\pi\big) - \pi \in [-\pi, \pi). \tag{2}$$

The image $x^\sharp$ can contain noise. In this paper, we do not aim at removing noise. If the unwrapping process is robust to the presence of noise, the noise can be removed after unwrapping using any image denoising method.

Because of the modulo operation, each pixel value $x^\sharp_{n_1,n_2}$ can be expressed as the sum of its wrapped version $y_{n_1,n_2} = (x^\sharp_{n_1,n_2})_w$ and an integer multiple of 2π. So, the goal is to recover all these integers. To that aim, we can notice that

$$(\nabla x^\sharp)_w = (\nabla y)_w, \tag{3}$$

where the discrete gradient ∇ is the concatenation of horizontal and vertical finite differences:

$$\nabla^h x^\sharp_{n_1,n_2} = x^\sharp_{n_1,n_2+1} - x^\sharp_{n_1,n_2}, \tag{4}$$

$$\nabla^v x^\sharp_{n_1,n_2} = x^\sharp_{n_1+1,n_2} - x^\sharp_{n_1,n_2}. \tag{5}$$

In 1-D, with $x^\sharp \in \mathbb{R}^N$, if the so-called Itoh condition [12] is satisfied, according to which every finite difference $x^\sharp_{n+1} - x^\sharp_n$ belongs to $[-\pi, \pi)$, then $x^\sharp$ can be recovered from y by integrating recursively, up to a global constant. Indeed, for every $n = 1, \dots, N-1$,

$$x^\sharp_{n+1} = x^\sharp_n + (y_{n+1} - y_n)_w. \tag{6}$$

This is the same in 2-D: if every horizontal and vertical finite difference of $x^\sharp$ belongs to $[-\pi, \pi)$, then

$$\nabla x^\sharp = (\nabla y)_w \tag{7}$$

and $x^\sharp$ can be recovered exactly from y, with the indeterminacy of the global constant resolved, for instance, by assuming that $x^\sharp_{1,1}$ is known or equal to $y_{1,1}$.

If the Itoh condition is not satisfied, for instance because of noise, with a slope or jump of absolute amplitude larger than π in at least one pixel of $x^\sharp$, the recovery is hopeless in 1-D without further assumptions, e.g. regularity of the second derivative [13]. In 2-D, however, an important property can be used to make unwrapping possible: for every $n_1 = 1, \dots, N_1 - 1$ and $n_2 = 1, \dots, N_2 - 1$,

$$\begin{aligned} & x^\sharp_{n_1+1,n_2+1} - x^\sharp_{n_1,n_2} & (8)\\ &= (x^\sharp_{n_1+1,n_2} - x^\sharp_{n_1,n_2}) + (x^\sharp_{n_1+1,n_2+1} - x^\sharp_{n_1+1,n_2}) & (9)\\ &= (x^\sharp_{n_1,n_2+1} - x^\sharp_{n_1,n_2}) + (x^\sharp_{n_1+1,n_2+1} - x^\sharp_{n_1,n_2+1}). & (10)\end{aligned}$$

In other words,

$$\nabla^h x^\sharp_{n_1,n_2} + \nabla^v x^\sharp_{n_1,n_2+1} = \nabla^v x^\sharp_{n_1,n_2} + \nabla^h x^\sharp_{n_1+1,n_2}. \tag{11}$$

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This property, sometimes called network flow constraint [14], holds for every image. It is the discrete equivalent of the property that the curl of the gradient field of a 2-D scalar function is zero. This property makes phase unwrapping much less ill-posed in 2-D than in 1-D. Thus, we can formulate phase unwrapping as a (nonconvex) optimization problem, expressed in terms of the finite differences of the reconstructed image: given some cost functions $f^h_{n_1,n_2}$ and $f^v_{n_1,n_2}$, which may or may not depend on n_1, n_2, h, v,

$$\underset{d^h,d^v}{\text{minimize}} \sum_{n_1,n_2} f^h_{n_1,n_2}(d^h_{n_1,n_2}) + f^v_{n_1,n_2}(d^v_{n_1,n_2}) \tag{12}$$

s.t. $(d^h_{n_1,n_2})_w = (\nabla^h y_{n_1,n_2})_w$, $(d^v_{n_1,n_2})_w = (\nabla^v y_{n_1,n_2})_w$, and $d^h_{n_1,n_2} + d^v_{n_1,n_2+1} = d^v_{n_1,n_2} + d^h_{n_1+1,n_2}$, for every n_1, n_2. Then, given the solution $(\tilde{d}^h, \tilde{d}^v)$, the reconstructed unwrapped image $\tilde{x}$, with $\nabla\tilde{x} = (\tilde{d}^h, \tilde{d}^v)$, is obtained by a simple raster-scan summation, like in (6). $\tilde{x}$ is a valid unwrapped image, in the sense that

$$(\tilde{x})_w = y. \tag{13}$$

When $f^h_{n_1,n_2} = f^v_{n_1,n_2}$ is the l_1 norm, the problem consists in minimizing the anisotropic total variation [15] of the reconstructed image, under the nonconvex constraint that its wrapped version is y. Bioucas-Dias and Valadão [16] proposed an algorithm called PUMA to solve this problem exactly, using graph cut techniques. It can be considered as the state of the art. Even better results can be expected by using a nonconvex cost function f_{n_1,n_2}. For instance, with $f(t) = \{0 \text{ if } t \in [-\pi, \pi),\ 1 \text{ else}\}$, the NP-hard problem is to minimize the number of adjacent pixel pairs, where the Itoh condition is not satisfied [17].

An alternative formulation is:

$$\underset{d^h,d^v}{\text{minimize}} \sum_{n_1,n_2} \sum_{s\in\{h,v\}} g\Big(d^s_{n_1,n_2} - (\nabla^s y_{n_1,n_2})_w\Big) \tag{14}$$

s.t. $d^h_{n_1,n_2} + d^v_{n_1,n_2+1} = d^v_{n_1,n_2} + d^h_{n_1+1,n_2}$, for every n_1, n_2. We can note that (14) is a particular case of (12), with $f^s_{n_1,n_2} = g(\cdot - (\nabla^s y_{n_1,n_2})_w)$, if $g(t) = +\infty$ whenever $t \notin 2\pi\mathbb{Z}$. If the latter condition is not met, there is no guarantee that the reconstructed image $\tilde{x}$ is a valid solution, since (13) may be violated. g can be chosen as a l_p norm [18]. If $p \geq 1$, the problem is convex and can be solved efficiently using modern proximal splitting techniques [19, 20, 21, 22]. For $p = 1$, minimum cost network flow techniques can be used, as proposed in [14]; we can note that the integer constraints $((d^h, d^v) - \nabla y)_w = 0$ are missing, but since the ℓ_1 norm induces sparsity, together with the network flow constraint, they will be satisfied [14].

Other types of methods have been proposed, including path following and branch cut methods [23, 8], belief propagation [24], Bayesian estimation [25], and spline approximation [13].

We will show in the next section how to solve the optimization problem (12) with *any* function f, by constructing a convex relaxation of it.

2. LIFTING AND CONVEX RELAXATION

Lifting consists in reformulating a difficult nonconvex optimization problem in a higher dimensional space. The lifted problem is still nonconvex but its combinatorial nature is unfolded to some extent. Thus, a convex relaxation of the lifted problem has a global minimizer, which will yield a good estimate of the solution to the initial problem, in general. This idea has been successfully applied to several segmentation and labeling problems in imaging and computer vision [26, 27, 28, 29, 30].

In our case, solving (12) amounts to find integers $k^h_{n_1,n_2}$ and $k^v_{n_1,n_2}$, for every n_1, n_2, $s \in \{h, v\}$, such that

$$d^s_{n_1,n_2} = (\nabla^s y_{n_1,n_2})_w + 2\pi k^s_{n_1,n_2}. \tag{15}$$

Every $k^s_{n_1,n_2}$ is assumed to lie in $-Q, \dots, Q$, for some known integer $Q \geq 1$. Let us define, for every $n_1 = 1, \dots, N_1 - 1$, $n_2 = 1, \dots, N_2 - 1$, the residual

$$r_{n_1,n_2} = \Big((\nabla^h y_{n_1,n_2})_w + (\nabla^v y_{n_1,n_2+1})_w \tag{16}$$

$$- (\nabla^v y_{n_1,n_2})_w - (\nabla^h y_{n_1+1,n_2})_w\Big)/(2\pi). \tag{17}$$

Then the network flow constraint is, for every n_1, n_2,

$$k^h_{n_1,n_2} + k^v_{n_1,n_2+1} - k^v_{n_1,n_2} - k^h_{n_1+1,n_2} = -r_{n_1,n_2}. \tag{18}$$

The lifting process consists in reformulating the problem by introducing, for every variable $k^s_{n_1,n_2}$, $s \in \{h, v\}$, a binary assignment vector $\mathbf{z}^s_{n_1,n_2} = (z^s_{n_1,n_2,q})^Q_{q=-Q}$ of size $2Q + 1$. The elements of a binary assignment vector are in $\{0, 1\}$ and their sum is 1. Then $k^s_{n_1,n_2}$ and $\mathbf{z}^s_{n_1,n_2}$ are related by

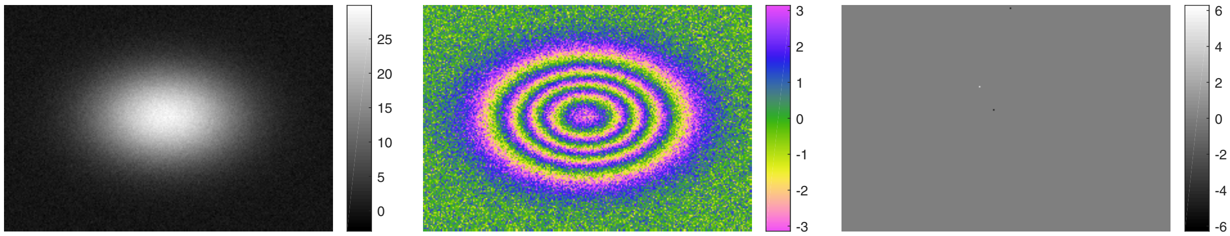
$$k^s_{n_1,n_2} = \textstyle\sum^Q_{q=-Q} q z^s_{n_1,n_2,q}. \tag{19}$$

Second, the cost function in (12) can be rewritten as

$$\sum_{n_1,n_2} \sum_{s\in\{h,v\}} \sum^Q_{q=-Q} c^s_{n_1,n_2,q} z^s_{n_1,n_2,q}, \tag{20}$$

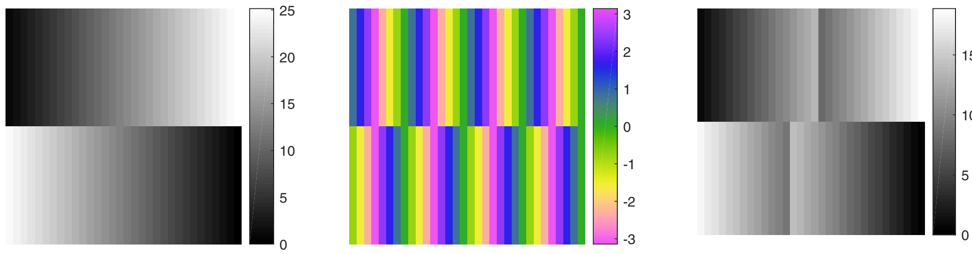
$$\text{with } c^s_{n_1,n_2,q} = f^s_{n_1,n_2}\big((\nabla^s y_{n_1,n_2})_w + 2\pi q\big). \tag{21}$$

We have all the ingredients to reformulate the problem (12) with only the vectors $\mathbf{z}^s_{n_1,n_2}$ as variables, by plugging (19) into (18). But since the convex relaxation of the lifted problem will simply consist in replacing the binary constraints by simplex constraints, the obtained convex relaxation would not be tight enough. For instance, the vector $(\frac{1}{2}, 0, \frac{1}{2})$ would play the same role as $(0, 1, 0)$ in the network flow constraint. We need to be more restrictive to ensure that the solution vectors will be binary. Therefore, we propose to further increase the dimension of the problem, by introducing not vectors but *matrices* $\mathbf{M}^1_{n_1,n_2}$ and $\mathbf{M}^2_{n_1,n_2}$, of size $(2Q + 1) \times (2Q + 1)$, for every $n_1 = 1, \dots, N_1 - 1$, $n_2 = 1, \dots, N_2 - 1$. Their elements are in $\{0, 1\}$ and their sum is one. The vectors $\mathbf{z}^s_{n_1,n_2}$



(a) original image $x^\sharp$ (b) wrapped image y (c) reconstruction error $\tilde{x} - x^\sharp$

Fig. 1. Experiment 1, see Sect. 3 for details. The unwrapped image $\tilde{x}$ with the proposed method, not shown, is visually identical to $x^\sharp$; in fact, their difference is zero at all but 3 pixels, as shown in (c). The result of PUMA is identical to ours. The result of COS is very similar, with an unwrapped image equal to $x^\sharp$ at all but 8 pixels.



(a) original image $x^\sharp$ (b) wrapped image y (c) unwrapped im., PUMA

Fig. 2. Experiment 2, see Sect. 3 for details. With COS and the proposed method, we have perfect reconstruction: $\tilde{x} = x^\sharp$.

will be retrieved as their marginals:

$$\textstyle\sum_{q=-Q}^{Q} M^1_{n_1,n_2,q,q'} = z^h_{n_1+1,n_2,q'}, \tag{22}$$

$$\textstyle\sum_{q'=-Q}^{Q} M^1_{n_1,n_2,q,q'} = z^v_{n_1,n_2,q}, \tag{23}$$

$$\textstyle\sum_{q=-Q}^{Q} M^2_{n_1,n_2,q,q'} = z^h_{n_1,n_2,q'}, \tag{24}$$

$$\textstyle\sum_{q'=-Q}^{Q} M^2_{n_1,n_2,q,q'} = z^v_{n_1,n_2+1,q}. \tag{25}$$

The network flow constraint can now be rewritten as:

$$\sum_{q,q'\,:\,q+q'=b} M^1_{n_1,n_2,q,q'} = \sum_{q,q'\,:\,q+q'=b-r_{n_1,n_2}} M^2_{n_1,n_2,q,q'}, \tag{26}$$

for every $b = -2Q, \dots, 2Q$ (an empty sum is set to zero). It is easy to rewrite the cost function in (12) as a summation over all indices of $\mathbf{M}^1$ and $\mathbf{M}^2$, similar to eq. (20).

Finally, a convex relaxation of this integer linear program is taken, by dropping the binary constraints. That is, $\mathbf{M}^1$ and $\mathbf{M}^2$ are assumed to lie in the simplex: their elements are nonnegative and their sum is one [31]. This linear program is solved using an overrelaxed Chambolle–Pock algorithm [20, 21]. If needed, a rounding step is performed as a postprocess on the solution to ensure that $(\tilde{x})_w = y$.

3. EXPERIMENTS

We compare Costantini's method [14], denoted by COS (using code found at https://mathworks.com/matlabcentral/fileexchange/25154-costantini-phase-unwrapping), the PUMA method [16] (using code on the first author's webpage), and the proposed method, for which we adopt the truncated ℓ_1 cost function defined as

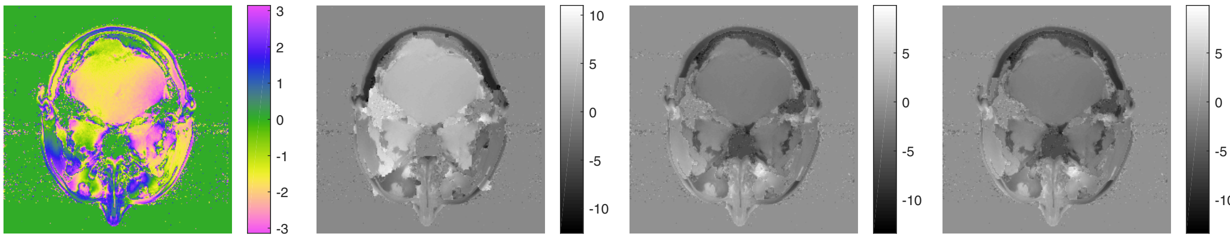
$$f : t \in \mathbb{R} \mapsto \min(|t|, \pi), \tag{27}$$

which has the advantage of being continuous and satisfying $f(-\pi) = f(\pi)$. This choice is arbitrary, and we leave for future work the comparison with other functions. In all experiments, the global solution of the problem (12) was achieved by our method. The MATLAB codes were run on an 2012 Apple Macbook Pro laptop with a 2.3 GHz CPU. The wrapped images in Figs. 1–4 were displayed using the C2 cyclic colormap designed by P. Kovesi [32].

Experiment 1. A 2-D Gaussian function with amplitude 9π was sampled in an image of size $N_1 = 176$, $N_2 = 256$. White Gaussian noise of std. dev. 0.7 was added. We used $Q = 1$. The reconstruction is almost perfect with all three methods, see in Fig. 1. The computation time of the COS, PUMA, proposed methods was about 1s, 1s, 5min, respectively.

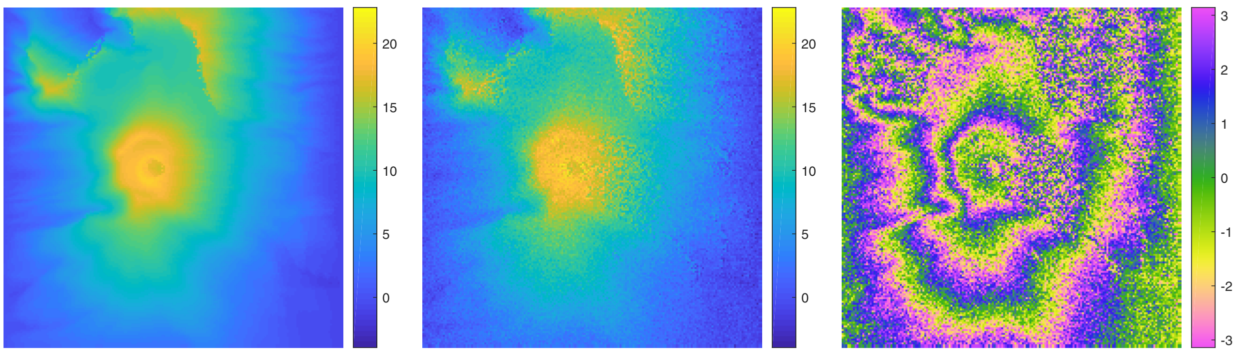
Experiment 2. A synthetic image, of size 32×32, or a shear of amplitude 8π was generated. Perfect reconstruction was achieved by COS and the proposed method ($Q = 4$). PUMA did not give the expected result; this shows the interest of a nonconvex cost function f in the presence of abrupt jumps.

Experiment 3. We consider the MRI head image from the freely available dataset of [4] (ftp://ftp.wiley.com/public/sci_

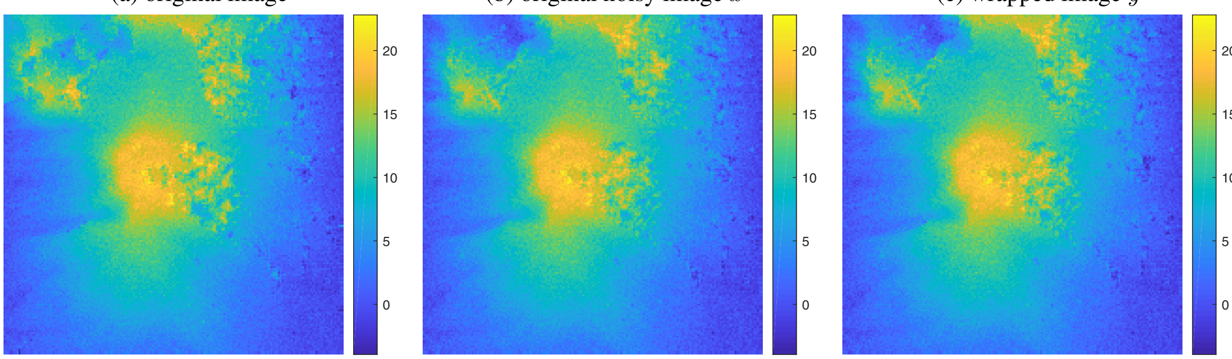


(a) wrapped image y (b) unwrapped image, COS (c) unwrapped image, PUMA (d) unwrapped im., proposed

Fig. 3. Experiment 3 with a MRI head image, see Sect. 3 for details.



(a) original image (b) original noisy image $x^\sharp$ (c) wrapped image y



(d) unwrapped image, COS (e) unwrapped image, PUMA (f) unwrapped image, proposed

Fig. 4. Experiment 4 with the elevation map of Mount Asama in Japan, see Sect. 3 for details.

tech_med/phase_unwrapping). There is no ground truth to compare the unwrapped images. COS gives an image with visible artifacts. PUMA and the proposed method yield comparable results.

Experiment 4. We consider the elevation map of Mount Asama, Japan, to simulate an InSAR acquisition. Noise following a realistic noise model [13] is added and the image is wrapped. COS gives an image with incorrect upper left part. PUMA and the proposed method yield similar results of good quality.

4. CONCLUSION

We have proposed a convex relaxation of the phase unwrapping problem based on lifting. That is, an equivalent nonconvex problem has been formulated in a higher dimensional space with 0-1 variables, then this lifted problem has been relaxed into a convex one. In future work, we plan to design new algorithms to solve such large-scale linear programs, by leveraging recent advances in probabilistic inference [33]. We also plan to compare the reconstruction quality for other costs than the truncated ℓ_1 cost considered here, for instance the truncated quadratic cost used in the Mumford–Shah model [34].

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